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Weighted estimates for powers and smoothing estimates of Schrödinger operators with inverse-square potentials

The Anh Bui [a,*], Piero D'Ancona [b], Xuan Thinh Duong [a], Ji Li [a], Fu Ken Ly [c]

[a] *Department of Mathematics, Macquarie University, NSW 2109, Australia*
[b] *Dipartimento di Matematica, Sapienza – Università di Roma, Piazzale A. Moro 2, 00185 Roma, Italy*
[c] *Mathematics Learning Centre, University of Sydney, NSW 2006, Australia*

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Abstract

Let $\mathcal{L}_a$ be a Schrödinger operator with inverse square potential $a|x|^{-2}$ on $\mathbb{R}^d$, $d \geq 3$. The main aim of this paper is to prove weighted estimates for fractional powers of $\mathcal{L}_a$. The proof is based on weighted Hardy inequalities and weighted inequalities for square functions associated to $\mathcal{L}_a$. As an application, we obtain smoothing estimates regarding the propagator $e^{it\mathcal{L}_a}$.

MSC: 35P25; 35Q55

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1. Introduction

In this paper, we consider the following Schrödinger operators with inverse-square potentials on $\mathbb{R}^d$, $d \geq 3$,

* Corresponding author.
E-mail addresses: the.bui@mq.edu.au, bt_anh80@yahoo.com (T.A. Bui), dancona@mat.uniroma1.it (P. D'Ancona), xuan.duong@mq.edu.au (X.T. Duong), ji.li@mq.edu.au (J. Li), ken.ly@sydney.edu.au (F.K. Ly).

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$$\mathcal{L}_a = -\Delta + \frac{a}{|x|^2} \quad \text{with} \quad a \geq -\Big(\frac{d-2}{2}\Big)^2. \tag{1}$$

Set

$$\sigma := \frac{d-2}{2} - \frac{1}{2}\sqrt{(d-2)^2 + 4a}.$$

The Schrödinger operator $\mathcal{L}_a$ is understood as the Friedrichs extension of $-\Delta + \frac{a}{|x|^2}$ defined initially on $C_c^\infty(\mathbb{R}^d\backslash\{0\})$. The condition $a \geq -\big(\frac{d-2}{2}\big)^2$ guarantees that $\mathcal{L}_a$ is nonnegative. It is well-known that $\mathcal{L}_a$ is self-adjoint and the extension may not be unique as $-\big(\frac{d-2}{2}\big)^2 < a < 1 - \big(\frac{d-2}{2}\big)^2$. For further details, we refer the readers to [22,23,20,29,32,25].

It is well known that Schrödinger operators with inverse-square potentials $\mathcal{L}_a$ have a wide range of applications in physics and mathematics spanning areas such as combustion theory, the Dirac equation with Coulomb potential, quantum mechanics and the study of perturbations of classic space–time metrics. See for example [4,5,33,20] and their references.

Recently there has been a spate of activity dedicated to the operator $\mathcal{L}_a$. Strichartz estimates, which are an effective tool for studying the behavior of solutions to nonlinear Schrödinger equations and wave equations related to $\mathcal{L}_a$, were investigated in [4,5]. In [16] the authors developed the study of Strichartz estimates for the propagators $e^{it(\Delta+V)}$ with $V(x) \sim |x|^{-2}$. The well-posedness and behavior of the solutions to the heat equation related to $\mathcal{L}_a$ was studied in [33]. In [34], using Morawetz-type inequalities and Sobolev norm properties related to $\mathcal{L}_a$, the long-time behavior of solutions to nonlinear Schrödinger equations associated to $\mathcal{L}_a$ was considered. More recently, the authors in [22] established the equivalence between L^p-based Sobolev norms defined in terms of $\mathcal{L}_a^{s/2}$ and in terms of $(-\Delta)^{s/2}$ for all regularities $0 < s < 2$.

In this paper, our first objective is to extend the estimates in [22] to weighted estimates. More precisely, we will prove the following result.

Theorem 1.1. *Suppose that $d \geq 3$, $a \geq -\Big(\frac{d-2}{2}\Big)^2$ and $0 < s < 2$. If $r_1 := 1 \vee \frac{d}{d-\sigma} < p < \frac{d}{(s+\sigma)\vee 0} := r_2$ (where $a \vee b = \max\{a, b\}$) with convention $\frac{d}{0} = \infty$ then for $w \in A_{p/r_1} \cap RH_{(r_2/p)'}$ we have*

$$\|(-\Delta)^{s/2} f\|_{L^p_w} \lesssim \|\mathcal{L}_a^{s/2} f\|_{L^p_w}. \tag{2}$$

If $1 < p < \infty$ with $p_1 := 1 \vee \frac{d}{d-\sigma} < p < \frac{d}{s\vee\sigma} := p_2$ then for $w \in A_{p/p_1} \cap RH_{(p_2/p)'}$ we have

$$\|\mathcal{L}_a^{s/2} f\|_{L^p_w} \lesssim \|(-\Delta)^{s/2} f\|_{L^p_w}. \tag{3}$$

The proof of the theorem relies heavily on the heat kernel of $\mathcal{L}_a$ in [28,24] (see Theorem 3.3) which is valid for $d \geq 3$. This is a main reason for the restriction $d \geq 3$.

Let us describe the motivation for the results in Theorem 1.1.

(i) When $s = 1$, (2) and (3) are known as the boundedness of the Riesz transforms and the reverse Riesz transforms, respectively. Note that the boundedness of the Riesz transforms related to $\mathcal{L}_a$ was obtained in [17]. Hence, Theorem 1.1 can be considered as a natural outgrowth of this direction of research.

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