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L^2 Solvability of boundary value problems for divergence form parabolic equations with complex coefficients

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Abstract

We consider parabolic operators of the form

$$\partial_t + \mathcal{L},\ \ \mathcal{L} = -\operatorname{div} A(X,t)\nabla,$$

in $\mathbb{R}^{n+2}_+ := \{(X,t) = (x, x_{n+1}, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} :\ x_{n+1} > 0\}$, $n \geq 1$. We assume that A is a $(n+1) \times (n+1)$-dimensional matrix which is bounded, measurable, uniformly elliptic and complex, and we assume, in addition, that the entries of A are independent of the spatial coordinate x_{n+1} as well as of the time coordinate t. For such operators we prove that the boundedness and invertibility of the corresponding layer potential operators are stable on $L^2(\mathbb{R}^{n+1}, \mathbb{C}) = L^2(\partial\mathbb{R}^{n+2}_+, \mathbb{C})$ under complex, L^∞ perturbations of the coefficient matrix. Subsequently, using this general result, we establish solvability of the Dirichlet, Neumann and Regularity problems for $\partial_t + \mathcal{L}$, by way of layer potentials and with data in L^2, assuming that the coefficient matrix is a small complex perturbation of either a constant matrix or of a real and symmetric matrix.

MSC: 35K20; 31B10

Keywords: Second order parabolic operator; Complex coefficients; Boundary value problems; Layer potentials; Kato problem

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1. Introduction and statement of main results

In this paper we study the solvability of the Dirichlet, Neumann and Regularity problems with data in L^2, in the following these problems are referred to as $(D2)$, $(N2)$ and $(R2)$, see (2.47) below for the exact definition of these problems, by way of layer potentials and for second order parabolic equations of the form

$$\mathcal{H}u := (\partial_t + \mathcal{L})u = 0, \tag{1.1}$$

where

$$\mathcal{L} := -\text{div } A(X,t)\nabla = -\sum_{i,j=1}^{n+1} \partial_{x_i}(A_{i,j}(X,t)\partial_{x_j}) \tag{1.2}$$

is defined in $\mathbb{R}^{n+2} = \{(X,t) = (x_1, .., x_{n+1}, t) \in \mathbb{R}^{n+1} \times \mathbb{R}\}$, $n \geq 1$. $A = A(X,t) = \{A_{i,j}(X,t)\}_{i,j=1}^{n+1}$ is assumed to be a $(n+1) \times (n+1)$-dimensional matrix with complex coefficients satisfying the uniform ellipticity condition

$$\begin{aligned}
&(i) \quad \Lambda^{-1}|\xi|^2 \leq \text{Re } A(X,t)\xi \cdot \bar{\xi} = \text{Re}\Big(\sum_{i,j=1}^{n+1} A_{i,j}(X,t)\xi_i\bar{\xi}_j\Big),\\
&(ii) \quad |A(X,t)\xi \cdot \zeta| \leq \Lambda|\xi||\zeta|,
\end{aligned} \tag{1.3}$$

for some Λ, $1 \leq \Lambda < \infty$, and for all $\xi, \zeta \in \mathbb{C}^{n+1}$, $(X,t) \in \mathbb{R}^{n+2}$. Here $u \cdot v = u_1v_1 + ... + u_{n+1}v_{n+1}$, $\bar{u}$ denotes the complex conjugate of u and $u \cdot \bar{v}$ is the standard inner product on $\mathbb{C}^{n+1}$. In addition, we consistently assume that

$$A(x_1, .., x_{n+1}, t) = A(x_1, .., x_n), \text{ i.e., } A \text{ is independent of } x_{n+1} \text{ and } t. \tag{1.4}$$

We study $(D2)$, $(N2)$ and $(R2)$ for the operator $\mathcal{H}$ in $\mathbb{R}^{n+2}_+ = \{(x, x_{n+1}, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : x_{n+1} > 0\}$, with data prescribed on $\mathbb{R}^{n+1} = \mathbb{R}^{n+2}_+ = \{(x, x_{n+1}, t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \ x_{n+1} = 0\}$. Assuming (1.3)–(1.4), as well as the De Giorgi–Moser–Nash estimates stated in (2.24)–(2.25) below, we first prove (Theorem 1.6, Corollary 1.7) that the solvability of $(D2)$, $(N2)$ and $(R2)$, by way of layer potentials, is stable under small complex L^∞ perturbations of the coefficient matrix. Subsequently, using Theorem 1.6, Corollary 1.7, we establish the solvability for $(D2)$, $(N2)$ and $(R2)$, by way of layer potentials, when the coefficient matrix is either

(i) a small complex perturbation of a constant (complex) matrix (Theorem 1.8), or

(ii) a real and symmetric matrix (Theorem 1.9), or

(iii) a small complex perturbation of a real and symmetric matrix (Theorem 1.10). (1.5)