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A weighted identity for stochastic partial differential operators and its applications ☆

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Abstract

In this paper, a weighted identity for some stochastic partial differential operators (with complex principal parts) is established. This identity presents a unified approach in establishing Carleman-type estimates for some deterministic/stochastic partial differential equations. Based on this identity, one can deduce some known global Carleman estimates for stochastic parabolic equations, stochastic Schrödinger equations, stochastic transport equations and their deterministic counterparts. Meanwhile, as its applications, we derive two different Carleman estimates for linear forward stochastic complex Ginzburg–Landau equations. They can be used to study the controllability/observability and inverse problems for some stochastic complex Ginzburg–Landau equations, respectively.

MSC: primary 93B05; secondary 93B07, 93C20

Keywords: Stochastic partial differential equation; Controllability; Observability; Inverse problem; Global Carleman estimate

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1. Introduction

This paper is devoted to a weighted identity for a class of stochastic partial differential operators. Based on this identity, one can derive global Carleman estimates for deterministic/stochastic partial differential equations of different type. This idea of proving Carleman-type estimates from a weighted identity first came from the Russian literature [13], in which some unique continuation results were established, based on suitable Carleman estimates. Carleman-type estimates are a class of energy estimates with exponential weights. They were originally introduced by T. Carleman in 1939 to prove a strong unique continuation property for some elliptic equations in [4]. Up to now, Carleman estimates have become a powerful tool in studying deterministic/stochastic partial differential equations, and the related control and inverse problems. For example, this type of weighted energy estimates was used to study the unique continuation property of partial differential equations (see [10,12] for example), the uniqueness and stability of Cauchy problems (see [2,3,8] for example), inverse problems (see [1,9,11] for example) and the controllability (see [6,7,21–23] and references therein), respectively.

In what follows, we give a simple example to introduce the basic idea of establishing global Carleman estimates by means of a pointwise weighted identity.

Example 1 *(A Carleman estimate for first order differential operators).* Let G be a nonempty open subset of $\mathbb{R}^n$ with a smooth boundary. For any given $\gamma_0 \in C(\overline{G})$ and $\gamma \in \left[C^1(\overline{G})\right]^n$, consider the following first order differential operator:

$$\mathcal{L}_0 u = \gamma \cdot \nabla u + \gamma_0 u \quad \text{in} \quad \overline{G}. \tag{1.1}$$

Set

$$\phi(x) = |x - x_0|^2, \qquad \text{for some } x_0 \in \mathbb{R}^n. \tag{1.2}$$

Then, we have the following known Carleman estimate for the operator $\mathcal{L}_0$ in (1.1).

Lemma 1.1. *Assume that for some $x_0 \in \mathbb{R}^n \setminus \overline{G}$ and a positive constant c_0,*

$$\gamma(x) \cdot (x - x_0) \le -c_0, \qquad \textit{in } \overline{G}. \tag{1.3}$$

Then there exist constants $\lambda^* > 0$ and $C > 0$, such that for any $\lambda \ge \lambda^*$,

$$\lambda \int_G e^{2\lambda\phi} u^2 dx \le C \int_G e^{2\lambda\phi} |\mathcal{L}_0 u|^2 dx, \quad \textit{for any } u \in C_0^1(G). \tag{1.4}$$

Proof of Lemma 1.1. For any parameter $\lambda > 0$, put

$$\ell(x) = \lambda\phi(x) \quad \text{and} \quad \theta = e^{\lambda\phi},$$

where ϕ is given in (1.2). Then it is easy to check that

$$(\theta^2 u)\gamma \cdot \nabla u = \theta^2 \gamma \cdot \nabla\Big(\frac{1}{2}u^2\Big) = \text{div}\Big(\frac{1}{2}\theta^2 u^2 \gamma\Big) - \theta^2\Big[\frac{1}{2}\text{div}\,\gamma + 2\lambda\gamma \cdot (x - x_0)\Big]u^2. \tag{1.5}$$