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Generalized solutions for inextensible string equations

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Abstract

We study the system of equations of motion for inextensible strings. This system can be recast into a discontinuous system of conservation laws as well as into the total variation wave equation. We prove existence of generalized Young measure solutions with non-negative tension after transforming the problem into a system of conservation laws and approximating it with a regularized system for which we obtain uniform estimates of the energy and the tension. We also discuss sufficient conditions for non-negativity of the tension for strong solutions.

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1. Introduction

An inextensible string is defined (cf. [4]) to be the one for which the stretch is constrained to be equal to 1, whatever system of forces is applied to it. As in [30], some authors refer to it as a *chain* which is a long but very thin material that is inextensible but completely flexible, and hence mathematically described as a rectifiable curve of fixed length. Dynamics of pipes, flagella, chains, or ribbons of rhythmic gymnastics, mechanism of whips, and galactic motion

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are only a few phenomena and applications that can be related to inextensible strings (see [10, 21,18] for more details).

The motion executed by a homogeneous, inextensible string with unit length and density can be modeled by the system

$$\begin{cases} \eta_{tt}(t,s) = \big(\sigma(t,s)\,\eta_s(t,s)\big)_s + g, \ \ s \in [0,1], \\ |\eta_s| = 1, \end{cases} \tag{1.1}$$

where $g \in \mathbb{R}^3$ is the given gravity vector, $\eta \in \mathbb{R}^3$ is the unknown position vector for material point s at time t. The unknown scalar multiplier σ, which is called *tension*, satisfies the equation

$$\sigma_{ss}(t,s) - |\eta_{ss}(t,s)|^2\sigma(t,s) + |\eta_{st}(t,s)|^2 = 0 \tag{1.2}$$

(see Section 2.4 for the derivation of (1.2) from (1.1)). We are given the initial positions and velocities of the string as

$$\eta(0,s) = \alpha(s) \ \text{ and } \ \eta_t(0,s) = \beta(s). \tag{1.3}$$

There are several options for boundary conditions:

a) two fixed ends:

$$\eta(t,0) = \alpha(0) \ \text{ and } \ \eta(t,1) = \alpha(1) \tag{1.4}$$

b) two free ends:

$$\sigma(t,0) = \sigma(t,1) = 0 \tag{1.5}$$

c) the “ring” or periodic conditions (here it is convenient to consider $s \in \mathbb{R}$ instead of $s \in [0,1]$):

$$\eta(t,s) = \eta(t,s+1) \ \text{ and } \ \sigma(t,s) = \sigma(t,s+1) \tag{1.6}$$

d) the “whip” boundary conditions when one end is free and one is fixed:

$$\sigma(t,0) = 0 \ \text{ and } \ \eta(t,1) = 0. \tag{1.7}$$

We make the convention that $s = 0$ corresponds to the free end while the end $s = 1$ is fixed at the origin of the space.

Even though the analysis of the dynamics of inextensible strings subject to different kinds of boundary conditions is a notable problem which goes back to Galileo, Leibniz and Bernoulli (cf. [30,4,26]), and it has been investigated by many authors in various contexts (see e.g. [10,22,21, 33,42]), there are still very few results about general well-posedness. According to [26], V. Yudovich was interested in this problem (possibly because of its relation to the Euler equations, see our Section 2.6), and obtained some unpublished results. One of the available existence results is by Reeken [31,32] who proves well-posedness for an infinite string with gravity when the initial data is near the trivial (downwards vertical) stable stationary solution (close in H^{26}).