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First and second order necessary conditions for stochastic optimal controls

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Abstract

The main purpose of this paper is to establish the first and second order necessary optimality conditions for stochastic optimal controls using the classical variational analysis approach. The control system is governed by a stochastic differential equation, in which both drift and diffusion terms may contain the control variable and the set of controls is allowed to be nonconvex. Only one adjoint equation is introduced to derive the first order necessary condition; while only two adjoint equations are needed to state the second order necessary conditions for stochastic optimal controls.

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1. Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space (satisfying the usual conditions), on which a 1-dimensional standard Wiener process $W(\cdot)$ is defined such that $\mathbb{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$ is the natural filtration generated by $W(\cdot)$ (augmented by all the P-null sets).

Let us consider the following controlled stochastic differential equation

$$\begin{cases} dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \quad t \in [0, T], \\ x(0) = x_0 \in K, \end{cases} \tag{1.1}$$

with the cost functional

$$J(u(\cdot), x_0) = \mathbb{E}\Big[\int_0^T f(t, x(t), u(t))dt + g(x(T))\Big]. \tag{1.2}$$

Here $u(\cdot)$ is the control variable with values in a closed nonempty subset U of $\mathbb{R}^m$ (for some fixed $m \in \mathbb{N}$), $x(\cdot)$ is the state variable with values in $\mathbb{R}^n$ (for some given $n \in \mathbb{N}$), K is a closed nonempty subset in $\mathbb{R}^n$, and $b, \sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}^n$, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}$ and $g : \mathbb{R}^n \times \Omega \to \mathbb{R}$ are given functions (satisfying suitable conditions to be stated later). As usual, when the context is clear, we omit the ω $(\in \Omega)$ argument in the defined functions.

Denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ respectively the inner product and norm in $\mathbb{R}^n$ or $\mathbb{R}^m$, which can be identified from the contexts, by $\mathcal{B}(X)$ the Borel σ-field of a metric space X, and by $\mathcal{U}_{ad}$ the set of $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable and $\mathbb{F}$-adapted stochastic processes with values in U such that $\mathbb{E}\int_0^T |u(t, \omega)|^2 dt < \infty$. Any $u(\cdot) \in \mathcal{U}_{ad}$ is called an admissible control, the corresponding state $x(\cdot; x_0)$ of (1.1) with initial datum $x_0 \in K$ is called an admissible state, and (x, u, x_0) is called an admissible triple. An admissible triple $(\bar{x}, \bar{u}, \bar{x}_0)$ is called optimal if

$$J(\bar{u}(\cdot), \bar{x}_0) = \inf_{\substack{u(\cdot) \in \mathcal{U}_{ad} \\ x_0 \in K}} J(u(\cdot), x_0). \tag{1.3}$$

The purpose of this paper is to establish first and second order necessary optimality conditions for problem (1.3). We refer to [4,5,16,21] and references cited therein for some early works on this subject. Although the stochastic optimal control theory was developing almost simultaneously with the deterministic one, its results are much less fruitful than those obtained for the deterministic control systems. The main reasons are due to some essential difficulties (or new phenomena) when the diffusion term of the stochastic control system depends on the control variable and the control region lacks convexity. In contrast with the deterministic case, for stochastic optimal control problems when spike variations are used as perturbations, the cost functional needs to be expanded up to the *second order* and *two adjoint equations* have to be introduced to derive the *first order necessary optimality conditions*. A stochastic maximum principle for this general case was established in [27]. On the other hand, to derive the second order necessary