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On nonlocal quasilinear equations and their local limits

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Abstract

We introduce a new class of quasilinear nonlocal operators and study equations involving these operators. The operators are degenerate elliptic and may have arbitrary growth in the gradient. Included are new nonlocal versions of p-Laplace, ∞-Laplace, mean curvature of graph, and even strongly degenerate operators, in addition to some nonlocal quasilinear operators appearing in the existing literature. Our main results are comparison, uniqueness, and existence results for viscosity solutions of linear and fully nonlinear equations involving these operators. Because of the structure of our operators, especially the existence proof is highly non-trivial and non-standard. We also identify the conditions under which the nonlocal operators converge to local quasilinear operators, and show that the solutions of the corresponding nonlocal equations converge to the solutions of the local limit equations. Finally, we give a (formal) stochastic representation formula for the solutions and provide many examples.

MSC: 35R09; 45K05; 35J60; 35J62; 35J70; 35K59; 47G20; 35D40; 35A01; 35B51; 35B40

Keywords: Nonlocal quasilinear equations; Infinity-Laplace; p-Laplace; Viscosity solutions; Well-posedness; Local limits

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1. Introduction

In this paper we introduce a new class of gradient dependent Lévy type diffusion operators and study the well-posedness, stability, and some asymptotic behavior of equations involving such operators. The operators we will consider are the following,

$$L[u, Du] = (L_1 + L_2)[u, Du]$$

where

$$L_1[u, Du](x) = \int_{\mathbb{R}^P} u\big(x + j_1(Du, z)\big) - u(x) - j_1(Du, z) \cdot Du(x)\, \mathrm{d}\mu_1(z)\,, \tag{1.1}$$

$$L_2[u, Du](x) = \int_{\mathbb{R}^P} u\big(x + j_2(Du, z)\big) - u(x)\, \mathrm{d}\mu_2(z)\,, \tag{1.2}$$

and μ_1, μ_2 are non-negative Lévy measures and j_1, j_2 are measurable functions (see Section 2). Here the strength and direction of the diffusion depend on the gradient, and hence as we explain below, these operators are natural generalizations of the local (non-divergence form) quasilinear operators

$$L_0(Du, D^2u) = \frac{1}{2}\mathrm{tr}\big(\sigma(Du)\sigma(Du)^T D^2u\big) + b(Du)Du.$$

The operators are allowed to degenerate ($j_1 = 0$ or $j_2 = 0$ in some set) and have arbitrary growth in the gradient, so ∞-Laplace, p-Laplace, and strongly degenerate operators are included. Included are also "explicit" operators of the form (cf. Section 3.2),

$$a(Du)\Big[-(-\Delta)^{\frac{\alpha}{2}}u\Big] \qquad \text{for all} \qquad \alpha \in (0,2) \ \text{ and } \ a \in C(\mathbb{R}^N; \mathbb{R}^+). \tag{1.3}$$

We want to study equations involving the operator L, and to simplify and focus on the new issues, the main part of this paper is devoted to the following special problem:

$$F\big(u, Du, L[u, Du]\big) = f(x) \qquad \text{in} \qquad \mathbb{R}^N, \tag{1.4}$$

where we assume F to be (degenerate) elliptic and strictly increasing in u (i.e. $D_u F > 0$). But for this equation, we make an effort to push for very general results. First we obtain comparison, uniqueness, stability, and existence results for bounded solutions of (1.4). These results are highly non-trivial due to the implicit nature of our operators and our weak integrability assumptions. Especially existence is very challenging as we discuss below. We then identify the limit problems where nonlocal operators converge to local ones,

$$L_\varepsilon[\phi, D\phi] \to L_0(D\phi, D^2\phi) \qquad \text{as} \qquad \varepsilon \to 0,$$

for any smooth and bounded function ϕ, and prove that the solutions u_ε of the corresponding nonlocal equations

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