J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Convergence rates for forward–backward dynamical systems associated with strongly monotone inclusions

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ARTICLE INFO

Article history:
Received 8 March 2016
Available online xxxx
Submitted by Helene Frankowska, Annamaria Barbagallo, and Gioconda Moscariello

Keywords:
Dynamical systems
Strongly monotone inclusions
Continuous forward–backward method
Convergence rates
Convex optimization problems

ABSTRACT

We investigate the convergence rates of the trajectories generated by implicit first and second-order dynamical systems associated to the determination of the zeros of the sum of a maximally monotone operator and a monotone and Lipschitz continuous one in a real Hilbert space. We show that these trajectories strongly converge with exponential rate to a zero of the sum, provided the latter is strongly monotone. We derive from here convergence rates for the trajectories generated by dynamical systems associated to the minimization of the sum of a proper, convex and lower semicontinuous function with a smooth convex one provided the objective function fulfills a strong convexity assumption. In the particular case of minimizing a smooth and strongly convex function, we prove that its values converge along the trajectory to its minimum value with exponential rate, too.

1. Introduction and preliminaries

The main topic of this paper is the investigation of convergence rates for implicit dynamical systems associated with monotone inclusion problems of the form

$$\text{find } x^* \in \mathcal{H} \text{ such that } 0 \in Ax^* + Bx^*, \tag{1}$$

where $\mathcal{H}$ is a real Hilbert space, $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone operator, $B : \mathcal{H} \to \mathcal{H}$ is a monotone and $\frac{1}{\beta}$-Lipschitz continuous operator for $\beta > 0$ and $A + B$ is ρ-strongly monotone for $\rho > 0$. Dynamical systems of implicit type have been already considered in the literature in [1,2,7,9,12,14–17].

We deal in a first instance with the first-order dynamical system with variable relaxation parameters

$$\begin{cases} \dot{x}(t) = \lambda(t)\left[J_{\eta A}\Big(x(t) - \eta B(x(t))\Big) - x(t)\right] \\ x(0) = x_0, \end{cases} \tag{2}$$

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[1] Research supported by FWF (Austrian Science Fund), Lise Meitner Programme, project M 1682-N25.

http://dx.doi.org/10.1016/j.jmaa.2016.07.007

where $x_0 \in \mathcal{H}$, $\lambda : [0, +\infty) \to [0, \infty)$ is a Lebesgue measurable function and $J_{\eta A}$ denotes the resolvent of the operator ηA for $\eta > 0$.

We notice that Abbas and Attouch considered in [1, Section 5.2] the dynamical system of the same type

$$\begin{cases} \dot{x}(t) + x(t) = \operatorname{prox}_{\mu\Phi}\big(x(t) - \mu B(x(t))\big) \\ x(0) = x_0 \end{cases} \tag{3}$$

in connection to the determination of the zeros of $\partial\Phi + B$, where $\Phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function, $B : \mathcal{H} \to \mathcal{H}$ is a cocoercive operator, $\partial\Phi$ denotes the convex subdifferential of Φ and $\operatorname{prox}_{\mu\Phi}$ denotes the proximal point operator of $\mu\Phi$.

Before that, Antipin in [7] and Bolte in [14] studied the convergence of the trajectories generated by

$$\begin{cases} \dot{x}(t) + x(t) = P_C\big(x(t) - \mu\nabla g(x(t))\big) \\ x(0) = x_0 \end{cases} \tag{4}$$

to a minimizer of the smooth and convex function $g : \mathcal{H} \to \mathbb{R}$ over the nonempty, convex and closed set $C \subseteq \mathcal{H}$, where $\mu > 0$ and P_C denotes the projection operator on the set C.

In the second part of the paper we approach the monotone inclusion (1) via the second-order dynamical system with variable damping and relaxation parameters

$$\begin{cases} \ddot{x}(t) + \gamma(t)\dot{x}(t) + \lambda(t)\Big[x(t) - J_{\eta A}\Big(x(t) - \eta B(x(t))\Big)\Big] = 0 \\ x(0) = u_0, \dot{x}(0) = v_0, \end{cases} \tag{5}$$

where $u_0, v_0 \in \mathcal{H}$, $\lambda : [0, +\infty) \to [0, \infty)$ and $\gamma : [0, +\infty) \to [0, \infty)$ are Lebesgue measurable functions, and $\eta > 0$.

Second-order dynamical systems of the form

$$\begin{cases} \ddot{x}(t) + \gamma\dot{x}(t) + x(t) - Tx(t) = 0 \\ x(0) = u_0, \dot{x}(0) = v_0, \end{cases} \tag{6}$$

for $\gamma > 0$ and $T : \mathcal{H} \to \mathcal{H}$ a nonexpansive operator, have been treated by Attouch and Alvarez in [8] in connection to the problem of approaching the fixed points of T (see also [11] and [3–6,10] for more on second-order dynamical systems).

For the minimization of the smooth and convex function $g : \mathcal{H} \to \mathbb{R}$ over the nonempty, convex and closed set $C \subseteq \mathcal{H}$, a continuous in time second-order gradient-projection approach has been considered in [7,8], having as starting point the dynamical system

$$\begin{cases} \ddot{x}(t) + \gamma\dot{x}(t) + x(t) - P_C(x(t) - \eta\nabla g(x(t))) = 0 \\ x(0) = u_0, \dot{x}(0) = v_0, \end{cases} \tag{7}$$

with constant damping parameter $\gamma > 0$ and constant step size $\eta > 0$.

For an exhaustive asymptotic analysis of the first and second-order dynamical systems (2) and (5), in case B is cocoercive, we refer the reader to [15] and [17], respectively. According to the above-named works, one can expect under mild assumptions on the relaxation and, in the second-order setting, on the damping functions, that the generated trajectories converge to a zero of $A + B$. The main scope of this paper is to show that when weakening the assumptions on B to monotonicity and Lipschitz continuity, however, provided that $A + B$ is strongly monotone, the trajectories converge strongly to the unique zero of $A + B$ with an exponential rate. Exponential convergence rates have been obtained also by Antipin in [7] for the