J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Self-contracted curves in Riemannian manifolds

Aris Daniilidis [a,*,1], Robert Deville [b,2], Estibalitz Durand-Cartagena [c,3], Ludovic Rifford [d,e,4]

[a] *DIM–CMM, UMI CNRS 2807, Beauchef 581, Torre Norte, piso 5, Universidad de Chile, Santiago 8370456, Chile*
[b] *Laboratoire Bordelais d'Analyse et Geométrie, Institut de Mathématiques de Bordeaux, Université de Bordeaux 1, 351 cours de la Libération, Talence Cedex 33405, France*
[c] *Departamento de Matemática Aplicada, ETSI Industriales, UNED, Juan del Rosal 12, Ciudad Universitaria, 28040 Madrid, Spain*
[d] *CMM, UMI CNRS 2807, Blanco Encalada 2120, Universidad de Chile, Santiago, Chile* [5]
[e] *Laboratoire J.A. Dieudonné, UMR CNRS 7351, Université Nice Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 2, France*

A R T I C L E I N F O

Article history:
Received 23 November 2016
Available online xxxx
Submitted by H. Frankowska

Keywords:
Self-contracted curve
Self-expanded curve
Rectifiable curve
Length
Secant
Riemannian manifold

A B S T R A C T

It is established that every self-contracted curve in a Riemannian manifold has finite length, provided its image is contained in a compact set.

Contents

* Corresponding author.
E-mail addresses: arisd@dim.uchile.cl (A. Daniilidis), Robert.Deville@math.u-bordeaux1.fr (R. Deville), edurand@ind.uned.es (E. Durand-Cartagena), ludovic.rifford@math.cnrs.fr (L. Rifford).
URLs: http://www.dim.uchile.cl/~arisd (A. Daniilidis), http://www.uned.es/personal/edurand (E. Durand-Cartagena), http://math.unice.fr/~rifford/ (L. Rifford).

[1] Research supported by the grants: BASAL PFB-03, FONDECYT 1130176, ECOS-CONICYT C14E06, REDES/CONICYT 150040 (Chile) and MTM2014-59179-C2-1-P (MINECO of Spain and ERDF of EU).
[2] Research supported by the grants: ECOS-SUD C14E06 (France) and REDES/CONICYT-150040 (Chile).
[3] Research supported by the grant MTM2015-65825-P (MINECO of Spain) and 2017-MAT12 (ETSI Industriales, UNED).
[4] Research supported by the grant REDES/CONICYT-150040 (Chile).
[5] Visiting Researcher.

http://dx.doi.org/10.1016/j.jmaa.2017.04.011
0022-247X/

1. Introduction

This work is devoted to the study of self-contracted curves on Riemannian manifolds $\mathcal{M}$.

Definition 1.1 *(Self-contracted curve)*. Let $\mathcal{M}$ be a Riemannian manifold and let d_g denote its geodesic distance. Given an interval $I = [0, T_\infty)$ with $T_\infty \in (0, \infty) \cup \{\infty\}$, a curve $\gamma : I \to \mathcal{M}$ is called *self-contracted*, if for every $t_1 \leq t_2 \leq t_3$ in I we have

$$d_g(\gamma(t_1), \gamma(t_3)) \geq d_g(\gamma(t_2), \gamma(t_3)). \tag{1.1}$$

In other words, for every $\tau \in [0, T_\infty)$ the function $t \mapsto d_g(\gamma(t), \gamma(\tau))$ is nonincreasing on $[0, \tau]$.

Self-contracted curves were introduced in [3, Definition 1.2.]. The motivation of this definition comes from the following example.

Example 1.2. If $f : \mathbb{R}^n \to \mathbb{R}^+$ is a $\mathcal{C}^1$-smooth convex function and if $\gamma : (0, +\infty) \to \mathbb{R}^n$ is smooth and satisfies $\gamma'(t) = -\nabla f\big(\gamma(t)\big)$ for all $t > 0$, then γ is a self-contracted curve.

Indeed, observe first that $\big(f(\gamma(t))\big)' = -\|\nabla f\big(\gamma(t)\big)\|^2 \leq 0$, thus the function $t \mapsto f\big(\gamma(t)\big)$ is nonincreasing. Therefore, since f is convex, if $\tau \geq t$, then

$$\frac{d}{dt}\Big(\frac{1}{2}\|\gamma(\tau) - \gamma(t)\|^2\Big) = \langle \gamma(\tau) - \gamma(t), \nabla f(\gamma(t)) \rangle \leq f(\gamma(\tau)) - f(\gamma(t)) \leq 0.$$

This proves that the function $t \mapsto \|\gamma(t) - \gamma(\tau)\|$ is nonincreasing on $[0, \tau]$.

One of the main interests in studying self-contracted curves lies in its applications. Rectifiability of self-contracted curves has been applied in different areas, including continuous and discrete dynamical systems, optimization and convergence of algorithms. See for example [3] and [4].

The definition of self-contractedness is purely metric: if φ is a nondecreasing function from an interval J onto I, then $\gamma \circ \varphi$ is also self-contracted, so this notion does not depend on the particular parametrization of the oriented graph $\{\gamma(t);\, t \in I\}$. Self-contractedness does not require prior smoothness or continuity assumption on the curve as shown by the following example.

Example 1.3. Let $\gamma : \mathbb{R} \to \mathbb{C}$ defined by $\gamma(t) = t$ if $t \leq -1$, $\gamma(t) = -t$ if $-1 < t \leq 0$ and $\gamma(t) = it$ if $t > 0$. The curve γ is self-contracted, is not smooth at $t = 0$, is discontinuous at $t = -1$, and moreover does not admit a continuous self-contracted extension, *i.e.* there exists no continuous self-contracted curve $\Gamma : \mathbb{R} \to \mathbb{C}$ such that $\{\Gamma(t) : t \in \mathbb{R}\} \supset \{\gamma(t) : t \in \mathbb{R}\}$.

In a Euclidean setting it has been established in [4, Section 3] (and independently in [8] for continuous curves) that bounded self-contracted curves have finite length. In both cases the proof was based on an old

Please cite this article in press as: A. Daniilidis et al., Self-contracted curves in Riemannian manifolds, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.04.011