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A remark on an integral characterization of the dual of BV

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A B S T R A C T

In this paper, we show how under the continuum hypothesis one can obtain an integral representation for elements of the topological dual of the space of functions of bounded variation in terms of Lebesgue and Kolmogorov–Burkill integrals.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^d$ be open and denote by $BV(\Omega)$ the set of functions of bounded variation in Ω, that is, those functions $u \in L^1(\Omega)$ with finite total variation:

$$|Du|(\Omega) := \sup_{\substack{\Phi \in C_c^1(\Omega;\mathbb{R}^d), \\ \|\Phi\|_{C_0(\Omega;\mathbb{R}^d)} \leq 1}} \int_\Omega u \operatorname{div} \Phi < +\infty. \tag{1.1}$$

When equipped with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega),$$

$BV(\Omega)$ is a Banach space, though in contrast to the Sobolev case, smooth compactly supported functions fail to be dense in the norm topology, nor is the space even separable. Its importance then lies in the fact that it is in some sense the natural space in a variety of problems in the calculus of variations where energies exhibit linear bounds in the gradient of the field – a notable example is in the study of minimal surfaces (see [5] for the systematic study of the space in this context).

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A key tool in its study is the compactness of bounded sets with respect to the weak-star convergence of measures provided by the Banach–Alaoglu theorem, as $BV(\Omega)$ can be embedded in the product space of integrable functions crossed with $\mathbb{R}^d$-valued finite Radon measures $L^1(\Omega) \times M_b(\Omega; \mathbb{R}^d)$. More than this, $BV(\Omega)$ is itself a dual space (a result we describe in Section 2 in more detail) for which one has an integral representation of the duality pairing. However, a problem which has not been clearly explained is that of giving an integral characterization of its dual. We were curious to see if this could be done, as the literature on the space itself has only partial known results. The problem can be dated to at least the 1977 paper of Meyers and Ziemer [9] on Poincaré–Wirtinger inequalities that provides a characterization of the positive measures in the dual of $BV(\Omega)$, while such a question is raised explicitly in the 1984 AMS summer meeting on Geometric Measure Theory and the Calculus of Variations (see [2], Problem 7.4 on p. 458). It is again stated in the 1998 paper of De Pauw [3], who gives an integral representation for the dual of the space of special functions of bounded variation. More recently, the problem has been discussed in the papers of Phuc and Torres [10] and Torres [11].

It turns out that one can give an integral representation of the dual of $BV(\Omega)$, and that the solution existed more or less before any reference to such a problem. Indeed, in the papers [7,8], assuming the continuum hypothesis, Mauldin characterized the dual of spaces of finite Radon measures in the case the set of all Radon measures on the space has cardinality at most $2^{\aleph_0}$. As a consequence of his result and the Hahn–Banach theorem, one can easily get the following integral representation for the dual of the $BV(\Omega)$. To this end, let us recall that given a function $u \in BV(\Omega)$, the measure derivative Du can be written as

$$Du = \nabla u \mathcal{L}^d + D^j u + D^c u,$$

where ∇u is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$, $D^j u = \nu_u (u^+ - u^-) \mathcal{H}^{d-1} \llcorner J_u$ with J_u a $(d-1)$-rectifiable set, and $D^c u = D^s u - D^j u$ for $D^s u$ the portion of Du which is singular with respect to the Lebesgue measure.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be open and $L \in (BV(\Omega))'$. Under ZFC set theory and the Continuum Hypothesis, there exists $g \in L^\infty(\Omega)$, $G_0 \in L^\infty(\Omega; \mathbb{R}^d)$, an $\mathcal{H}^{d-1}$ bounded and measurable function $G_1 : \Omega \to \mathbb{R}^d$ and $\Psi : \mathcal{B}(\Omega) \to \mathbb{R}$ a bounded, Borel set function for which*

$$L(u) = \int_\Omega g u \, dx + \int_\Omega G_0 \cdot \nabla u \, dx + \int_{J_u} G_1 \cdot dD^j u + (K) \int \Psi \cdot dD^c u$$

for all $u \in BV(\Omega)$. Here, the last integral on the right hand side is understood in the Kolmogorov–Burkill sense. Conversely, any such integral functional is in the dual of $BV(\Omega)$.

Here we observe that the integral representation is in terms of both Lebesgue and Kolmogorov–Burkill integrals. We recall that given a set function $\psi : \mathcal{B}(\Omega) \to \mathbb{R}$ and a countably additive set function $\mu : \mathcal{B}(\Omega) \to [-\infty, \infty]$, the Kolmogorov–Burkill integral is defined as follows. We say that ψ is Kolmogorov–Burkill integrable with respect to μ if there exists a number $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a finite partition of Ω into Borel sets D such that if D' is any refinement of D,

$$\left| \sum_{B \in D'} \psi(B)\mu(B) - I \right| < \varepsilon.$$

As discussed by Kolmogorov in [6], this notion of integration encompasses both Lebesgue and Riemann integration. Indeed, the Kolmogorov integral generalizes the idea of Leibniz on areas as set functions, a sort of Riemann integral for the set function $\psi \cdot \mu$. Notice that while μ is countably additive, ψ satisfies no such property. Moreover, for any fixed $\mu \in M_b(\Omega)$, the Kolmogorov–Burkill integral projects onto a Lebesgue