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Very weak solutions to elliptic equations with singular convection term ☆

Luigi Greco, Gioconda Moscariello, Gabriella Zecca *

Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi di Napoli "Federico II", Via Cintia, 80126 Napoli, Italy

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ABSTRACT

We study Dirichlet problem for a nonlinear equation with a drift term. Despite the presence of the singular convection term, we establish existence and uniqueness of a solution in spaces larger than the natural one.

1. Introduction

We consider the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(\mathcal{A}(x, \nabla u) + \mathcal{B}(x, u)\right) = \operatorname{div} F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a (regular) bounded domain of $\mathbb{R}^N$, $N > 2$ and $F \in L^p(\Omega, \mathbb{R}^N)$, $1 < p \leqslant 2$.

We assume that $\mathcal{A}\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$ the following structure conditions:

$$|\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)| \leqslant \beta |\xi - \eta| \tag{1.2}$$

$$\alpha |\xi - \eta|^2 \leqslant \langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \rangle \tag{1.3}$$

$$\mathcal{A}(x, 0) = 0 \tag{1.4}$$

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* Corresponding author.
E-mail addresses: luigreco@unina.it (L. Greco), gmoscari@unina.it (G. Moscariello), g.zecca@unina.it (G. Zecca).

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where $0 < \alpha < \beta$ are positive constants. The vector field $\mathcal{B}\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function verifying:

(i) There exists a nonnegative function $b\colon \Omega \to \mathbb{R}_+$ in the *weak-L^N space*, $b \in L^{N,\infty}(\Omega)$, such that

$$|\mathcal{B}(x,s) - \mathcal{B}(x,t)| \leqslant b(x)|s-t|, \tag{1.5}$$

for a.e. $x \in \Omega$ and for any $s, t \in \mathbb{R}$.

(ii)

$$\mathcal{B}(x,0) \equiv 0 \text{ a.e. in } \Omega. \tag{1.6}$$

Given $F \in L^p(\Omega, \mathbb{R}^N)$, $p \geqslant 1$, a distributional solution to Problem (1.1) is a function $u \in W_0^{1,p}(\Omega)$ verifying the integral identity

$$\int_\Omega \langle \mathcal{A}(x, \nabla u) + \mathcal{B}(x,u), \nabla\varphi\rangle \, \mathrm{d}x = \int_\Omega \langle F, \nabla\varphi\rangle \, \mathrm{d}x \tag{1.7}$$

for all $\varphi \in C_0^\infty(\Omega)$. Note that if $p < 2$ the functional

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla u\rangle \, \mathrm{d}x$$

could be unbounded.

In the evolution case, the model of the homogeneous equation in (1.1) is the Fokker Planck equation. For recent applications to game theory see [16] and references therein.

When b is equal to zero, that is, the lower order term is not present, there is a wide literature concerning existence of distributional solutions to (1.1), when the right hand side term in the equation is a positive Borel measure starting by G. Stampacchia [19] and L. Boccardo–T. Gallouet [2].

Since the example by Serrin [18], it is known that uniqueness generally fails in $W_0^{1,p}(\Omega)$ for p far from 2. On the other hand, if $F \in L^p(\Omega, \mathbb{R}^N)$, existence and uniqueness of solution to problem (1.1) is given in [4,6,7] provided p is close to 2.

If $b(x)$ does not vanish identically, the problem is in general not coercive. In view of Sobolev embedding theorem, a natural condition in order to study Problem (1.1) is $b \in L^{N,\infty}(\Omega)$. However, this assumption does not guarantee existence of a solution to (1.1), even in the linear case (see Section 4). Assuming b in the Lebesgue space $L^N(\Omega)$ or, more generally, in the Lorentz space $L^{N,q}(\Omega)$, $N < q < \infty$, problem (1.1) has been studied in [3,14] for linear case and in [20] for nonlinear case, under the assumption that div F belongs to $L^1(\Omega)$. Other related results can be found in [9,17].

In all these papers a distributional solution is obtained combining an a priori estimate on the superlevel set of a solution, due to Boccardo [3], with a suitable approximation argument.

When div$F$ is not integrable or is not a measure, even in the case $b \equiv 0$, the main problem consists in getting a priori estimates. Indeed, test functions whose gradient is essentially proportional to gradient of u are not admissible.

For this reason, we use the terminology introduced by Iwaniec and Sbordone in [13], referring to such distributional solutions also as very weak solutions.

In this paper we show that a condition to find very weak solutions to Problem (1.1) whenever p is close to 2, is that b lies in a suitable subset of $L^{N,\infty}(\Omega)$. More precisely, our result reads

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