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Necessary second-order conditions for a weak local minimum in a problem with endpoint and control constraints ☆

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A B S T R A C T

One of the first steps towards necessary second-order optimality conditions in problems with constraints was taken by Dubovitskii and Milyutin in 1965. They offered a scheme that was very effective in smooth optimization problems, but seemed to be not suitable for applications in problems with pointwise control constraints. In this article we consider a modification of the Dubovitskii–Milyutin scheme, which allows to derive necessary second-order conditions for a weak local minimum in optimal control problems with a finite number of endpoint constraints of equality and inequality type and with pointwise control constraints of inequality type given by smooth functions. Assuming that the gradients of active control constraints are linearly independent, we provide rather straightforward proof of these conditions for a measurable and essentially bounded optimal control.

1. Introduction

In this paper, we discuss necessary second-order conditions for a weak local minimum in optimal control. There is an extensive literature on the subject. Starting from 70-ies, important results on second-order conditions were obtained in Russia, in particular, by the Milyutin school in Moscow [12]. Almost at the same time there were articles of J. Warga [20,21]. They followed by contributions of F. Bonnans, H. Frankowska, H. Maurer, K. Malanowski, V. Zeidan and many others. Among recent publications let us mention [1–3,10,11,14–18]. More detailed historical commentary and bibliographical notes can be found in, e.g., [1,11,12,14,17,18].

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This paper can be considered as a continuation of paper [16], where a simplified version of sufficient second-order conditions in optimal control [14] was presented with straightforward proofs. Now our aim is to give relatively simple proofs of *necessary* second-order conditions in an optimal control problem with endpoint and control constraints. These proofs go back to the method of critical directions, proposed by Dubovitskii and Milyutin in [9, Sec. 2], which perfectly worked in the mathematical programming and calculus of variations (see, e.g., [17]), but seemed to be insufficient in optimal control. It was one of the reasons for creating the abstract theory of higher order conditions [12], which allowed to derive both necessary and sufficient optimality conditions in different classes of nonsmooth optimization problems with constraints, including problems of optimal control. The abstract theory was used several times by A.A. Milyutin and his co-workers in order to obtain new optimality conditions. In particular, no-gap necessary and sufficient second-order conditions for extremals with discontinuous controls (which took into account variations of the discontinuity points of the reference control), in optimal control problems with regular mixed state-control constraints of inequality and equality type, were obtained by the author, see e.g., [14,15]. For singular extremals, such conditions were obtained by A.V. Dmitruk, see, e.g., [5,6]. But the proofs, presented in these works (based on [12]), are very lengthy and complicated. Mostly this is due to the generality and completeness of the results.

In the present work, we will show, that a certain modification of the method of critical directions [9] allows to derive second-order necessary conditions not only in smooth problems with constraints, but also in optimal control problems with control constraints of the form $g(u) \le 0$, for which we assume that the gradients $g_i'(u)$ of the active constraints are *linearly independent*. The proofs, presented in this paper, are much shorter and straightforward than those in [15], but certainly the setting of the problem and the obtained results are less general. Our conditions have the form of the requirement of positive semi-definiteness of an associated quadratic form (or a maximum of quadratic forms taken over the set of normed tuples of Lagrange multipliers) on the so-called critical cone. Here, we do not take into account variations of switching points of the control (if there are any), as it was done in [15]. We obtain necessary second-order conditions for a weak local minimum at the point $(\bar{x}, \bar{u})$ with an arbitrary measurable and essentially bounded control $\bar{u}$. We do not make any assumptions which refer to the whole system of constraints at the point $(\bar{x}, \bar{u})$, such as Mangasarian–Fromovitz constraint qualification.

Earlier, in our paper with F. Bonnans [3], we obtained second-order necessary conditions for a weak local minimum in an optimal control problem with *positively independent* gradients of control constraints (see [3, Theorem 5.2]), using the Cominetti result [4]. But some additional restrictive assumptions, concerning the whole system of constraints (see in [3] qualification hypotheses (61) and (80)), were required. The same can be said about assumptions in paper [2], devoted to problems with mixed and state constraints.

Recently, H. Frankowska proposed a new, rather straightforward method for obtaining second-order conditions, which concerned the Mayer optimal control problem involving an *arbitrary closed* control set $U \subset \mathbb{R}^m$. In our joint publication [10], using second order tangents to U, we have shown, that if $\bar{u}$ is an optimal control, then an associated quadratic functional should be nonnegative for all elements in the second order jets to U along $\bar{u}$. It left open the question of how to adapt this method for problems with a finite number of additional endpoint inequality constraints. In some sense, the present publication is a step in this direction, since its proofs are also rather straightforward, and the control problem involves a finite number of endpoint constraints of inequality and equality type, although, as it was said above, the closed control set U is not arbitrary.

The paper is organized as follows. In Section 2, we set an optimal control problem with endpoint and control constraints and formulate, in Theorem 1, the first- and second-order necessary conditions for a weak local minimum in this problem. The proof of Theorem 1 is given in Section 3. Carrying out the proof, we define the operator G of equality constraints and consider the two possible cases for the derivative $G'(\bar{w})$ of this operator at the reference point $\bar{w} = (\bar{x}, \bar{u})$: 1) the case when $G'(\bar{w})$ is not surjective (the irregular case) and 2) the case when it is surjective (the regular case). In the first case Theorem 1 holds trivially, since