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State constrained optimal control problems with time delays

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ABSTRACT

In a recent, related, paper, necessary conditions in the form of a Maximum Principle were derived for optimal control problems with time delays in both state and control variables. Different versions of the necessary conditions covered fixed end-time problems and, under additional hypotheses, free end-time problems. These conditions improved on previous conditions in the following respects. They provided the first fully non-smooth Pontryagin Maximum Principle for problems involving delays in both state and control variables, only special cases of which were previously available. They provide a strong version of the Weierstrass condition for general problems with possibly non-commensurate control delays, whereas the earlier literature does so only under structural assumptions about the dynamic constraint. They also provided a new 'two-sided' generalized transversality condition, associated with the optimal end-time. This paper provides an extension of the Pontryagin Maximum Principle of the earlier paper for time delay systems, to allow for the presence of a unilateral state constraint. The new results fully recover the necessary conditions of the earlier paper when the state constraint is absent, and therefore retain all their advantages but in a setting of greater generality.

1. Introduction

In an earlier paper [2], first order optimality conditions, in the form of a Maximum Principle, were derived for an optimal control problem, in which we seek to minimize a cost

$$J(x(.),u(.)) \;=\; g(x(S),x(T)) + \int_{[S,T]} L(t,x(t-h_0),\ldots,x(t-h_N),u(t-h_0),\ldots,u(t-h_N))dt\,,$$

over control functions $u(.)$ such that $u(t) \in U(t)$, a.e., and state trajectories $x(.)$ satisfying specified boundary conditions and a dynamic constraint, formulated as a controlled delay differential equation:

$$\dot{x}(t) = f(t,x(t-h_0),\ldots,x(t-h_N),u(t-h_0),\ldots,u(t-h_N)), \quad \text{a.e. } t \in [S,T]\,. \tag{1.1}$$

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Here, $[S,T]$ is a given time interval, $h_0 < h_1 < \ldots < h_N$ are given numbers such that $h_0 = 0$, $f(.\,.) : [S,T] \times \mathbb{R}^{(1+N)\times n} \times \mathbb{R}^{(1+N)\times m} \to \mathbb{R}^n$ and $L(.\,.) : [S,T] \times \mathbb{R}^{(1+N)\times n} \times \mathbb{R}^{(1+N)\times m} \to \mathbb{R}$ are given functions and $U(t)$, $S \le t \le T$, and C are given sets. We write $h := h_N$. Notice that, according to this formulation, delays may occur in both x and u variables. [2] also provided necessary conditions for the related free end-time problem. These optimality conditions were announced in [3].

This paper generalizes the necessary conditions of the earlier paper, relating to both fixed and free end-time problems, to allow for a pure state constraint, of the form

$$\psi(t, x(t)) \le 0 \text{ for all } t \in [S,T]\,, \tag{1.2}$$

in which $\psi(.,.) : [S,T] \times \mathbb{R}^n \to \mathbb{R}$ is a given scalar valued function. The unrestrictive nature of the hypotheses we shall impose on $\psi(.)$ will mean that a wide variety of formulations of the state constraint (for example, multiple functional inequality constraints and implicit constraints $x(t) \in A(t)$) is covered by our theory.

Maximum Principles are necessary conditions asserting the existence of a costate arc $p(.)$ which, together with minimizing state trajectory and control $(\bar{x}(.), \bar{u}(.))$ under consideration, have various properties. In the standard case, when the dynamic constraint takes the form of a controlled differential equation, $p(.)$ satisfies a differential equation. The distinctive feature of optimal control problems involving time delays (but without state constraints) is that $p(.)$ now satisfies (in the case $L(\ldots) \equiv 0$) an advance-differential equation of the type:

$$\begin{aligned} -\dot{p}(t) \;=\; \sum_{k=0}^{N} p(t+h_k) \cdot \nabla_{x_k} f(t+h_k, \bar{x}(t-h_0+h_k), \ldots, \bar{x}(t-h_N+h_k)),\\ \bar{u}(t-h_0+h_k), \ldots, \bar{u}(t-h_N+h_k)) \;\text{ a.e. } t \in [S,T]\,, \end{aligned} \tag{1.3}$$

in which $\nabla_{x_k} f(\ldots)$ refers to the derivative of $f(t, x_0, \ldots, x_k, \ldots, x_N, u_0, \ldots, u_N)$ w.r.t. x_k. When a state constraint (1.2) is included in the formulation of the optimal control problem, the co-state equation takes the modified form:

$$\begin{aligned} -\dot{p}(t) \;=\; \sum_{k=0}^{N} q(t+h_k) \cdot \nabla_{x_k} f(t+h_k, \bar{x}(t-h_0+h_k), \ldots, \bar{x}(t-h_N+h_k)),\\ \bar{u}(t-h_0+h_k), \ldots, \bar{u}(t-h_N+h_k)) \;\text{ a.e. } t \in [S,T]\,, \end{aligned} \tag{1.4}$$

expressed in terms of the bounded variation function

$$q(t) := p(t) + \int_{[S,t]} \nabla_x \psi(s, \bar{x}(s)) \mu(ds).$$

Here, $\mu(.)$ is a non-decreasing bounded variation scalar valued function $\mu(.)$ on $[S,T]$, which is interpreted as the Lagrange multiplier associated with the state constraint. In non-smooth versions of the necessary conditions, (1.3) and (1.4) are replaced by differential inclusions expressed in terms of appropriate set-valued 'subdifferentials' $\partial_{x_k} f(\ldots)$ in place of $\nabla_{x_k} f(\ldots)$, or having related forms.

There is a long history of research into necessary conditions having the features described above. Early work of this nature [1,12,14,20] was based on the application of abstract multiplier rules due to Neustadt [16], Hestenes [13] and Warga [19,21]. There is an extensive Russian literature, based on 'Boltyanski approximating cones', quasi-convex families of arcs and similar constructs [9], a notable example being the 2005 paper [15] of Kharatishvili and Tadumadze, which can be seen as a culmination of work of the Russian school in this area, dating back to the late 1950's. Necessary conditions for problems with time delays,