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The discrete twofold Ellis–Gohberg inverse problem ☆

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A B S T R A C T

In this paper a twofold inverse problem for orthogonal matrix functions in the Wiener class is considered. The scalar-valued version of this problem was solved by Ellis and Gohberg in 1992. Under reasonable conditions, the problem is reduced to an invertibility condition on an operator that is defined using the Hankel and Toeplitz operators associated to the Wiener class functions that comprise the data set of the inverse problem. It is also shown that in this case the solution is unique. Special attention is given to the case that the Hankel operator of the solution is a strict contraction and the case where the functions are matrix polynomials.

1. Introduction

To state our main problem we need some notation and terminology about Wiener class functions. Throughout $\mathcal{W}^{n\times m}$ denotes the space of $n \times m$ matrix functions with entries in the Wiener algebra on the unit circle. Thus a matrix function φ belongs to $\mathcal{W}^{n\times m}$ if and only if φ is continuous on the unit circle and its Fourier coefficients $\ldots\varphi_{-1}, \varphi_0, \varphi_1, \ldots$ are absolutely summable. We set

$$\begin{aligned}
\mathcal{W}_+^{n\times m} &= \{\varphi \in \mathcal{W}^{n\times m} \mid \varphi_j = 0, \quad \text{for } j = -1, -2, \ldots\},\\
\mathcal{W}_-^{n\times m} &= \{\varphi \in \mathcal{W}^{n\times m} \mid \varphi_j = 0, \quad \text{for } j = 1, 2, \ldots\},\\
\mathcal{W}_d^{n\times m} &= \{\varphi \in \mathcal{W}^{n\times m} \mid \varphi_j = 0, \quad \text{for } j \neq 0\},\\
\mathcal{W}_{+,0}^{n\times m} &= \{\varphi \in \mathcal{W}^{n\times m} \mid \varphi_j = 0, \quad \text{for } j = 0, -1, -2, \ldots\},\\
\mathcal{W}_{-,0}^{n\times m} &= \{\varphi \in \mathcal{W}^{n\times m} \mid \varphi_j = 0, \quad \text{for } j = 0, 1, 2, \ldots\}.
\end{aligned}$$

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Given $\varphi \in \mathcal{W}^{n\times m}$ the function φ^* is defined by $\varphi^*(\zeta) = \varphi(\zeta)^*$ for each $\zeta \in \mathbb{T}$. Thus the j-th Fourier coefficient of φ^* is given by $(\varphi^*)_j = (\varphi_{-j})^*$. The map $\varphi \mapsto \varphi^*$ defines an involution which transforms $\mathcal{W}^{n\times m}$ into $\mathcal{W}^{m\times n}$, $\mathcal{W}_+^{n\times m}$ into $\mathcal{W}_-^{m\times n}$, $\mathcal{W}_{-,0}^{n\times m}$ into $\mathcal{W}_{+,0}^{m\times n}$, etc.

The data of the inverse problem we shall be dealing with consist of four functions, namely

$$\alpha \in \mathcal{W}_+^{p\times p}, \quad \beta \in \mathcal{W}_+^{p\times q}, \quad \gamma \in \mathcal{W}_-^{q\times p}, \quad \delta \in \mathcal{W}_-^{q\times q}, \tag{1.1}$$

and we are interested in finding $g \in \mathcal{W}_+^{p\times q}$ such that

$$\alpha + g\gamma - e_p \in \mathcal{W}_{-,0}^{p\times p} \quad \text{and} \quad g^*\alpha + \gamma \in \mathcal{W}_{+,0}^{q\times p}; \tag{1.2}$$

$$\delta + g^*\beta - e_q \in \mathcal{W}_{+,0}^{q\times q} \quad \text{and} \quad g\delta + \beta \in \mathcal{W}_{-,0}^{p\times q}. \tag{1.3}$$

Here e_p and e_q denote the functions identically equal to the identity matrices I_p and I_q, respectively. If g has these properties, we refer to g as a *solution to the twofold EG inverse problem associated with the data set* $\{\alpha, \beta, \gamma, \delta\}$. If a solution exists, then we know from Theorem 1.2 in [10] that necessarily the following identities hold:

$$\alpha^*\alpha - \gamma^*\gamma = a_0, \quad \delta^*\delta - \beta^*\beta = d_0, \quad \alpha^*\beta = \gamma^*\delta. \tag{1.4}$$

Here a_0 and d_0 are the zero-th Fourier coefficient of α and δ, respectively, and we identify the matrices with a_0 and d_0 with the matrix functions on $\mathbb{T}$ that are identically equal to a_0 and d_0, respectively. Our main problem is to find additional conditions that guarantee the existence of a solution and to obtain explicit formulas for a solution.

The EG inverse problem related to (1.2) only and using α and γ only has been treated in [11]. Here we deal with the inverse problem (1.2) and (1.3) together, and for that reason we refer to the problem as a twofold EG inverse problem. The acronym EG stands for R. Ellis and I. Gohberg, the authors of [1], where the inverse problem is solved for the scalar case, see [1, Section 4].

Given a data set $\{\alpha, \beta, \gamma, \delta\}$ and assuming both matrices a_0 and d_0 are invertible, our main theorem (Theorem 4.1) gives necessary and sufficient conditions in order that the twofold EG inverse problem associated with the given data set has a solution. Furthermore, we show that the solution is unique and we give an explicit formula for the solution in terms of the given data. The results obtained can be seen as an addition to Chapter 11 in the Ellis and Gohberg book [2]. For some more insight in the role of the matrices a_0 and d_0 in (1.4) we refer to Appendix A.

To understand better the origin of the problem and to prove our main results we shall restate the twofold EG inverse problem as an operator problem. This requires some further notation and terminology. For any positive integer n we denote by $\ell_+^2(\mathbb{C}^n)$ and $\ell_-^2(\mathbb{C}^n)$ the Hilbert spaces

$$\ell_+^2(\mathbb{C}^n) = \left\{ \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \mid \sum_{j=0}^{\infty} \|x_j\|^2 < \infty \right\}, \ell_-^2(\mathbb{C}^n) = \left\{ \begin{bmatrix} \vdots \\ x_{-2} \\ x_{-1} \\ x_0 \end{bmatrix} \mid \sum_{j=0}^{\infty} \|x_{-j}\|^2 < \infty \right\}. \tag{1.5}$$

We shall also need the corresponding ℓ^1-spaces which appear when the superscripts 2 in (1.5) are replaced by 1. Since an absolutely summable sequence is square summable, $\ell_\pm^1(\mathbb{C}^n) \subset \ell_\pm^2(\mathbb{C}^n)$. In the sequel the one column matrices of the type appearing in (1.5) will be denoted by

$$\begin{bmatrix} x_0 & x_1 & x_2 & \cdots \end{bmatrix}^\top \text{ and } \begin{bmatrix} \cdots & x_{-2} & x_{-1} & x_0 \end{bmatrix}^\top, \text{ respectively,}$$

with the $\top$-superscript indicating the block transpose. We will also use this notation when the entries are matrices. Finally, let h and k be the linear maps defined by