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On the univalence of polyharmonic mappings

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ABSTRACT

A $2p$-times continuously differentiable complex valued function $f = u + iv$ in a simply connected domain Ω is *polyharmonic* (or *p-harmonic*) if it satisfies the *polyharmonic* equation $\triangle^p F = 0$. Every polyharmonic mapping f can be written as $f(z) = \sum_{k=1}^{p} |z|^{2(p-1)} G_{p-k+1}(z)$, where each G_{p-k+1} is harmonic. In this paper we investigate the univalence of polyharmonic mappings on linearly connected domains and the relation between univalence of $f(z)$ and that of $G_p(z)$. The notion of stable univalence and logpolyharminc mappings are also considered.

1. Introduction

A $2p$-times continuously differentiable complex valued function $f = u + iv$ in a simply connected domain Ω is *polyharmonic* (or p-harmonic) if it satisfies the *polyharmonic* equation

$$\triangle^p f = 0,$$

where $p \geq 1$ is an integer and $\triangle$ represents the Laplacian operator

$$\triangle := 4\frac{\partial^2}{\partial z \partial \overline{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

and

$$\triangle^p f = \underbrace{\triangle ... \triangle}_{p} f = \triangle^{p-1} \triangle f.$$

Clearly when $p = 1$, f is harmonic and when $p = 2$, f is biharmonic. The properties of univalent harmonic mappings have been studied by many authors (see [7,16–19,21–23]). One the most fundamental articles on univalent harmonic mappings is due to Clunie–Shiel and Small ([16]).

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Biharmonic mappings and their univalence have been investigated recently by several authors (see [1,3–5,10]). Biharmonic functions arise in many physical situations, in fluid dynamics and elasticity problems which have many applications in engineering and biology (see [9,20,24,26,27]). In addition biharmonic mappings are closely related to the theory of Laguerre minimal surfaces ([6]). More recently, properties of polyharmonic functions are investigated. We refer to [11–14,25,28] for many interesting results on polyharmonic mappings.

If $\Omega \subset \mathbb{C}$ is a simply connected domain, then it is easy to see that (see [11]) every polyharmonic mapping f can be written as

$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z),$$

where each G_{p-k+1} is harmonic, that is $\triangle G_{p-k+1} = 0$ for $k \in \{1, ..., p\}$. This is known as the Almansi expansion (see [8]).

Throughout we consider polyharmonic functions defined on the unit disk $\mathbb{D} = \{z : |z| < 1\}$.

Definition 1. A domain $\Omega \subset \mathbb{C}$ is linearly connected if there exists a constant $M < \infty$ such that any two points $w_1, w_2 \in \Omega$ are joined by a path γ, $\gamma \subset \Omega$, of length $\ell(\gamma) \leq M|w_1 - w_2|$.

Such a domain is necessarily a Jordan domain, and for piecewise smoothly bounded domains, linear connectivity is equivalent to the boundary having no inward-pointing cusps.

In [15], Chuaqui and Hermandez, considered the relationship between the harmonic mapping $f = h + \overline{g}$ and its analytic factor h on linearly connected domains. They show that if h is an analytic univalent function, then every harmonic mapping $f = h + \overline{g}$ with dilatation $|\omega| < C$ is univalent if and only if $h(\mathbb{D})$ is linearly connected.

In [2], Abdulhadi and El Hajj showed analogous results for biharmonic functions. In this paper we generalize these results for polyharmonic mappings. Moreover some results are obtained for logpolyharmonic mappings. Recently properties of logbiharmonic and logpolyharmonic mappings have been investigated (see [11,25]). Many physical problems are modeled by logbiharmonic mappings particularly those arising from fluid flow theory.

The property of stable univalence is also considered.

2. Main results

A complex valued function $f : \Omega \to \mathbb{C}$ is said to belong to the class $C^1(\Omega)$ if $\Re f$ and $\Im f$ have continuous first order partial derivatives in Ω. We denote the Jacobian of f by

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2.$$

We also denote

$$\lambda_f = |f_z| - |f_{\overline{z}}|,$$

and

$$\Lambda_f = |f_z| + |f_{\overline{z}}|.$$

We then have

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