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Adjoint of the Toeplitz operator with the singular inner function

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ABSTRACT

Let ψ_{δ_1} be the singular inner function associated with the unit point mass at $z = 1$. We shall give an infinite dimensional closed subspace M of $\ker T^*_{\psi_{\delta_1}}$, the kernel of the adjoint of the Toeplitz operator $T_{\psi_{\delta_1}}$, satisfying that for every nonzero v in M, the smallest backward shift invariant subspace of H^2 containing v coincides with $\ker T^*_{\psi_{\delta_1}}$.

1. Introduction

Let H^2 be the Hardy space on the open unit disk $\mathbb{D}$ with variable z and L^2 the space of square integrable functions on the unit circle $\mathbb{T}$ with respect to the Lebesgue measure on $\mathbb{T}$. For a function f in H^2, we identify f with its boundary function on $\mathbb{T}$, so we also think of H^2 as $H^2 \subset L^2$. A function θ in H^2 is said to be inner if $|\theta| = 1$ a.e. on $\mathbb{T}$. We identify an inner function θ with $c\theta$ for every $c \in \mathbb{C}$ with $|c| = 1$, so $\theta_1 \neq \theta_2$ means $\theta_1 H^2 \neq \theta_2 H^2$.

We denote by P the orthogonal projection on L^2 onto H^2. For $\varphi \in L^\infty$, the space of bounded functions in L^2, the Toeplitz operator on H^2 is defined by $T_\varphi f = P(\varphi f), f \in H^2$. Put $\ker T_\varphi = \{f \in H^2 : T_\varphi f = 0\}$. We have $T^*_\varphi = T_{\overline{\varphi}}$, and $\ker T^*_\theta = H^2 \ominus \theta H^2$ for every inner function θ. A closed subspace N of H^2 is said to be backward shift invariant if $T^*_z N \subset N$. In this case, $T_z(H^2 \ominus N) \subset H^2 \ominus N$, and if $N \neq H^2$, then by the Beurling theorem there is an inner function θ satisfying $H^2 \ominus N = \theta H^2$. For a subset M of H^2, we denote by $[M]_*$ the smallest backward shift invariant subspace containing M. If $[M]_* \neq H^2$, then there

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is an inner function ζ_M satisfying that $H^2 \ominus [M]_* = \zeta_M H^2$. When $M = \{v\}$, we write $[M]_* = [v]_*$ and $H^2 \ominus [v]_* = \zeta_v H^2$. See [3,5,7] for the study of Toeplitz operators.

For a sequence $\{a_n\}_{n\geq 1}$ in $\mathbb{D}$ with $\sum_{n\geq 1}(1-|a_n|) < \infty$, the associated Blaschke product is defined by

$$b(z) = \prod_{n=1}^{\infty} \frac{-\overline{a}_n}{|a_n|} \frac{z-a}{1-\overline{a}_n z}, \quad z \in \mathbb{D}.$$

For a positive singular measure μ on $\mathbb{T}$, the associated singular function is defined by

$$\psi_\mu(z) = \exp\Big(- \int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(e^{i\theta}) \Big), \quad z \in \mathbb{D}.$$

It is well known that both b and ψ_μ are inner functions. See [6] for the function theory on $\mathbb{D}$.

In [2], Cowen and Gallardo-Gutiérrez studied the invariant subspace problem and Toeplitz operators (see also [1]). They posed three questions. In this paper, we answer two questions. Let δ_1 be the unit point mass at $z = 1$. We actually show that there is an infinite dimensional closed subspace M of $\ker T^*_{\psi_{\delta_1}}$ satisfying the following:

(i) $\zeta_v = \zeta_M = \psi_{\delta_1}$ for every nonzero $v \in M$.
(ii) Either $M \cap \ker T^*_\zeta = \{0\}$ or $M \cap \ker T^*_\zeta = M$ for every inner function ζ.

This answers Question 3 given in [2, p. 502] affirmatively and gives a counterexample for Question 2 given in [2, p. 501].

2. Examples

For each $\alpha \in \mathbb{D}$, put

$$b_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}, \quad z \in \mathbb{D}.$$

For inner functions θ_1, θ_2, we write $\theta_1 \prec \theta_2$ if θ_1 divides θ_2 in the family of inner functions.

Proposition 2.1. *Let M be a closed subspace of H^2 such that $[M]_* \neq H^2$ and $\dim M = \infty$. If ζ_M has a Blaschke factor, then there exists a nonzero $v_0 \in M$ satisfying the following.*

(i) *$\zeta_v \neq \zeta_M$ and $\zeta_v \prec \zeta_M$ for every nonzero $v \in M$ with $v \perp v_0$ and $\zeta_{v_0} = \zeta_M$.*
(ii) *$\{0\} \neq M \cap \ker T^*_{\zeta_v} \neq M$ for every nonzero $v \in M$ with $v \perp v_0$.*

Proof. (i) By the assumption, we may write $\zeta_M = b_\alpha \sigma$ for some $\alpha \in \mathbb{D}$ and some inner function σ. Since $\dim M = \infty$, σ is non-constant. By the definition of ζ_M, we have

$$M \not\subset H^2 \ominus \sigma H^2 \quad \text{and} \quad M \not\perp \sigma H^2 \ominus b_\alpha \sigma H^2 = \mathbb{C} \cdot \frac{\sigma}{1-\overline{\alpha}z}.$$

Let v_0 be the projection of $\sigma/(1-\overline{\alpha}z)$ onto M. Then $v_0 \in M$ and $v_0 \neq 0$. Let $v \in M$ with $v \perp v_0$. We have

$$v \perp \frac{\sigma}{1-\overline{\alpha}z} \quad \text{and} \quad v \perp \sigma H^2.$$

Hence $\zeta_v \prec \sigma \prec \zeta_M$, so $\zeta_v \neq \zeta_M$.