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Gradient estimates via the Wolff potentials for a class of quasilinear elliptic equations

Fengping Yao *, Min Zheng

Department of Mathematics, Shanghai University, Shanghai 200444, China

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In this paper we obtain the pointwise gradient estimates via the nonlinear Wolff potentials for weak solutions of a class of non-homogeneous quasilinear elliptic equations with measure data.

1. Introduction

In this paper we are mainly concerned with the pointwise gradient estimates for weak solutions of a class of non-homogeneous quasilinear elliptic equations with measure data

$$\mathrm{div}\ (a\left(|Du|\right)Du) = \mu \quad \text{in } \Omega, \tag{1.1}$$

where μ is a Borel measure with finite mass and $a : (0, \infty) \to (0, \infty) \in C^1(0, \infty)$ satisfies

$$0 \leq i_a =: \inf_{t>0} \frac{ta'(t)}{a(t)} \leq \sup_{t>0} \frac{ta'(t)}{a(t)} =: s_a < \infty. \tag{1.2}$$

We define

$$g(t) = ta(t) \tag{1.3}$$

* Corresponding author.
E-mail address: yfp@shu.edu.cn (F. Yao).

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and

$$G(t)=\int_0^t g(\tau)\,d\tau=\int_0^t \tau a(\tau)\,d\tau \quad \text{for } t\geq 0. \tag{1.4}$$

From (1.2) we observe that

$$g(t) \text{ is strictly increasing and continuous over } [0,+\infty), \tag{1.5}$$

and then

$$G(t) \text{ is increasing over } [0,+\infty) \text{ and strictly convex with } G(0)=0. \tag{1.6}$$

For example we see that

$$G(t)=t^p \quad \text{and} \quad G(t)=t^p\log\left(1+t\right) \quad \text{for any } p\geq 2 \tag{1.7}$$

satisfy the condition (1.2). Especially when $a(t)=t^{p-2}$ (and then $G(t)=t^p/p$), (1.1) is reduced to the p-Laplacian equation

$$\operatorname{div}\left(|Du|^{p-2}\,Du\right)=\mu \quad \text{for } p\geq 2.$$

Many authors [7–9,13,14,18,19,22,28,30,32] have extensively studied regularity estimates for p-Laplacian elliptic equation and the general case. Moreover, Cianchi and Maz'ya [10,11] proved global Lipschitz regularity for the Dirichlet and Neumann elliptic boundary value problems of

$$\operatorname{div}\left(a\left(|Du|\right)Du\right)=f \tag{1.8}$$

with the condition (1.2). Furthermore, Cianchi and Maz'ya [12] obtained a sharp estimate for the decreasing rearrangement of the length of the gradient for the Dirichlet and Neumann elliptic boundary value problems of (1.8) with (1.2) and

$$Ct^{p-1}\leq ta(t)\leq C(t^{p-1}+1) \quad \text{for any } t>0 \text{ and } p\in[2,n).$$

The pointwise estimates of the weak solution u via the Wolff potential $W^{\mu}_{\beta,p}(x,R)$ for nonlinear elliptic equations with right-hand side measure are developed by [20,21,31]. We recall that the classical non-linear Wolff potential is defined by

$$W^{\mu}_{\beta,p}(x,R):=\int_0^R\left(\frac{|\mu|\left(B(x,\varrho)\right)}{\varrho^{n-\beta p}}\right)^{\frac{1}{p-1}}\frac{d\varrho}{\varrho} \quad \text{for } \beta\in\left(0,\frac{n}{p}\right].$$

Furthermore, the classical case of general p-Laplacian equations (even with coefficients $a(x,\cdot)$ depending on the space variable x) was treated before by Mingione [29] in the case that $p=2$, and Duzaar and Mingione [15,17] in the case that $p\neq 2$. In the singular subquadratic case that $p<2$ even a linear gradient potential estimate was established by Duzaar and Mingione [16], i.e. the nonlinear Wolff-potential is replaced by $I_1^{|\mu|}(x,2R)^{\frac{1}{p-1}}$, where

$$I_1^{|\mu|}(x,R)=\int_0^R\frac{|\mu|(B(x,\varrho))}{\varrho^{n-1}}\frac{d\varrho}{\varrho} \quad \text{and} \quad |\mu|(B(x,\varrho))=\int_{B(x,\varrho)}|\mu|dy.$$