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The time dependent Maxwell system with measurable coefficients in Lipschitz domains ☆

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ABSTRACT

Semigroup theory is employed to study inhomogeneous boundary conditions for the time-dependent Maxwell equations with anisotropic material parameters in a non-smooth domain. Only boundedness and measurability of the coefficients are assumed, and the boundary data is assumed to be time independent. When the material parameters are positive definite, a higher order Sobolev theory is developed for the time-dependent Maxwell system.

1. Introduction

In the Maxwell system of electromagnetism, the material parameters ε and μ are characteristic quantities which determine the propagation of light within a specified material. In an anisotropic medium, ε and μ are 3×3 positive definite matrices, with entries depending on position [9]. This note is devoted to solving the full time-dependent Maxwell equations with variable ε and μ and nonzero boundary data in a bounded, Lipschitz domain in $\mathbb{R}^3$. We also consider the case of constant, positive definite ε and μ, and develop a higher order Sobolev theory in this case. We give a rigorous analysis of function spaces related to the Maxwell system in this setting, and prove useful characterizations of these spaces.

The guiding motivation for this problem is an electromagnetic inverse problem, originating in the work of A. P. Calderón in [3]. The electromagnetic Dirichlet-to-Neumann map is the boundary map which sends the tangential component of the electric field to the tangential component of the magnetic field on the boundary,

$$\Lambda : \nu \times \mathbf{E}|_{\partial\Omega} \mapsto \nu \times \mathbf{H}|_{\partial\Omega}$$

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The inverse problem is as follows: given two sets of material parameters (ε_1, μ_1) and (ε_2, μ_2), assume that the corresponding maps $\Lambda_1 = \Lambda_2$ everywhere on the boundary. Then one hopes to prove that this map uniquely determines the material parameters, i.e. $\varepsilon_1 \equiv \varepsilon_2$ and $\mu_1 \equiv \mu_2$. To solve the inverse problem, one first needs to solve the forward (Cauchy) problem, which is what we establish in this paper. There is a vast literature on this subject in the time harmonic regime, but what we wish to mention is that the time dependent case in this general setting (i.e. a non-smooth domain) to our knowledge has not yet been studied. When the domain is of class C^2, though, various boundary conditions for the time dependent problem (perfect conductor and Silver–Müller) are analyzed in [17], with results similar to our Theorem 4.1.

The outline of this paper is as follows. In Section 2 we discuss the necessary function spaces for the Cauchy problem. Then in Section 3 we formulate the Maxwell equations as an abstract evolution equation in a Hilbert space. We then proceed in Section 4 to solve the homogeneous boundary value problem. Section 5 is our main focus, where we impose nonzero boundary data on a Lipschitz domain and solve the resulting initial boundary value problem; the main result is Theorem 5.3. A key tool in the proof is a surjectivity result of Tartar [19]. Finally, we discuss in Section 6 how to extend the previous results to higher order Sobolev spaces if the material parameters are positive definite matrices with constant entries.[1] The main result in this section for the Maxwell system is Theorem 6.4.

We would like to thank the referee for pointing out the reference [17].

2. Function spaces for the Maxwell problem

Before going into the details of the problem, we find it useful here to discuss some of the function spaces that we will use. For the following, Ω will denote a bounded, Lipschitz domain in $\mathbb{R}^3$. We recall:

Definition 2.1. A bounded domain $\Omega \subset \mathbb{R}^3$ is called a *Lipschitz domain* if for each point $p \in \partial\Omega$, there exists an open set $\mathcal{O} \subset \mathbb{R}^3$ such that $p \in \mathcal{O}$, and an orthogonal coordinate system with coordinates $\xi = (\xi_1, \xi_2, \xi_3)$ having the following property:

There exists a vector $b \in \mathbb{R}^3$ so that

$$\mathcal{O} = \{\xi : -b_j < \xi_j < b_j,\ 1 \le j \le 3\}$$

and a Lipschitz continuous function ϕ defined on the set

$$\mathcal{O}' = \{\xi' \in \mathbb{R}^2 : -b_j < \xi_j < b_j,\ 1 \le j \le 2\}$$

such that

$$\Omega \cap \mathcal{O} = \{\xi : \xi_3 < \phi(\xi'),\ \xi' \in \mathcal{O}'\}$$

and

$$\partial\Omega \cap \mathcal{O} = \{\xi : \xi_3 = \phi(\xi'),\ \xi' \in \mathcal{O}'\}$$

Recall that $L^2(\Omega)$ denotes the Lebesgue space of square integrable functions on Ω, and $\left(L^2(\Omega)\right)^3$ denotes the space of square integrable vectors $f = (f_1, f_2, f_3)$. Further recall that the L^2 scalar product of $U = (u_1, u_2, u_3)$ and $V = (v_1, v_2, v_3)$ is given by

$$(U, V)_{L^2(\Omega)^3} := \int_\Omega \sum_{j=1}^{3} u_j \overline{v}_j dx$$

[1] Such parameters frequently arise in applications, e.g. in crystal optics [14].