J. Math. Anal. Appl. 452 (2017) 970–989

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Pointwise selection theorems for metric space valued bivariate functions

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ARTICLE INFO

Article history:
Received 14 February 2017
Available online 22 March 2017
Submitted by B. Bongiorno

Keywords:
Functions of several variables
Metric space
Total joint variation
Pointwise convergence
Selection principle

ABSTRACT

We introduce a pseudometric TV on the set M^X of all functions mapping a rectangle X on the plane $\mathbb{R}^2$ into a metric space M, called the total joint variation. We prove that if two sequences $\{f_j\}$ and $\{g_j\}$ of functions from M^X are such that $\{f_j\}$ is pointwise precompact on X, $\{g_j\}$ is pointwise convergent on X with the limit $g \in M^X$, and the limit superior of $\mathrm{TV}(f_j, g_j)$ as $j \to \infty$ is finite, then a subsequence of $\{f_j\}$ converges pointwise on X to a function $f \in M^X$ such that $\mathrm{TV}(f, g)$ is finite. One more pointwise selection theorem is given in terms of total ε-variations ($\varepsilon > 0$), which are approximations of the total variation as $\varepsilon \to 0$.

1. Main results

Pointwise selection principles are existence theorems guaranteeing the existence of a pointwise convergent subsequence of a given sequence of functions. The historically first example is the classical Helly's Theorem [25], [32, Section VIII.4]: *a uniformly bounded sequence of real monotone functions on a closed interval* $[a, b]$ *in* $\mathbb{R}$ *contains a pointwise convergent subsequence whose limit is a bounded monotone function on* $[a, b]$. As a corollary, the monotonicity of functions may be replaced by the *uniform boundedness* of their *Jordan's variations*. A far reaching consequence of the latter result is (Theorem C below and) the existence of selections of bounded (generalized) variation of univariate multifunctions of bounded (generalized) variation whose values are compact subsets of a metric space [10].

The purpose of this paper is to provide pointwise selection theorems for functions of several variables valued in an arbitrary metric space. In order to present the results in a simple and principal form and avoid (unnecessary) technicalities, we consider the case of bivariate functions on a closed rectangle.

We begin with reviewing definitions and facts needed for our results.

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http://dx.doi.org/10.1016/j.jmaa.2017.03.053

Given two points $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, we write $x < y$ (or $x \le y$) provided $x_1 < y_1$ and $x_2 < y_2$ (or $x_1 \le y_1$ and $x_2 \le y_2$, respectively), and we denote by $I_x^y = \{z \in \mathbb{R}^2 : x \le z \le y\} = [x_1, y_1] \times [x_2, y_2]$ the rectangle in $\mathbb{R}^2$ with the end-points x and y. In what follows, points a, $b \in \mathbb{R}^2$, $a < b$, are fixed, and the domain of bivariate functions is the rectangle I_a^b.

Recall that a function $\nu : I_a^b \to \mathbb{R}$ is said to be *totally monotone* if, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in I_a^b$ with $x \le y$, we have

$$\nu(y_1, a_2) - \nu(x_1, a_2) \ge 0, \qquad \nu(a_1, y_2) - \nu(a_1, x_2) \ge 0, \text{ and}$$
$$\nu(x_1, x_2) - \nu(y_1, x_2) - \nu(x_1, y_2) + \nu(y_1, y_2) \ge 0.$$

Totally monotone functions are well-studied [1,4,6,23,26–30] (they are called *positively monotonely monotone* in [26, III.4.3]). We recall the following two results for totally monotone functions, also needed below.

Theorem A *([26, III.5.4], [34]). The points of discontinuity of a totally monotone function on I_a^b lie on at most a countable collection of lines parallel to the coordinate axes in $\mathbb{R}^2$.*

Theorem B *(Helly's selection principle from [7], [26, III.6.5]). A uniformly bounded sequence of totally monotone functions on I_a^b contains a subsequence, which converges pointwise on I_a^b to a bounded totally monotone function.*

There are a number of extensions of Theorem B for multivariate functions of bounded variation in various senses: [26,27,30,31] for real valued functions, and [5,19–22] for *metric semigroup* valued functions (see below).

Of main interest in this paper are *metric space* valued functions on I_a^b. Our approach to the pointwise selection theorems for (sequences of) such functions is based on two notions of pseudometrics, the *joint increment* and *joint mixed difference*, to be defined as follows.

Let X be a nonempty set (in the sequel, X is a closed interval $I = [a, b]$ in $\mathbb{R}$, or the rectangle I_a^b in $\mathbb{R}^2$), (M, d) be a metric space with metric d, and M^X be the set of all functions $f : X \to M$ mapping X into M. Given $f \in M^X$ and $u \in M$, we set $f_u(x) = d(u, f(x))$ for all $x \in X$ (so that f_u maps X into $[0, \infty)$) and note that

$$d(f(x), g(y)) = \max_{u \in M} |f_u(x) - g_u(y)| \text{ for all } f, g \in M^X \text{ and } x, y \in X. \tag{1.1}$$

In particular, setting $(f - g)_u(x) = f_u(x) - g_u(x)$ for $u \in M$ and $x \in X$, we find

$$d(f(x), g(x)) = \max_{u \in M} |(f - g)_u(x)|. \tag{1.2}$$

Although the 'subtraction' $f - g$ is given by $(u, x) \mapsto (f - g)_u(x)$ and maps $M \times X$ into $\mathbb{R}$, passing to $h = f - g$ and $h_u(x) = f_u(x) - g_u(x)$, for the sake of brevity, will be a convenient tool in some proofs below.

The *joint increment* of two functions $f, g \in M^X$ on the two-point set $\{x, y\} \subset X$ is (the increment of $f - g$, i.e.) the quantity introduced in [15, Chapter 5] and [16, Section 2] by

$$\begin{aligned} |(f, g)(x, y)| &= \sup_{u \in M} |(f - g)_u(x) - (f - g)_u(y)| \\ &= \sup_{u \in M} \big| d(u, f(x)) - d(u, f(y)) - d(u, g(x)) + d(u, g(y)) \big|. \end{aligned} \tag{1.3}$$

Now suppose $X = I = [a, b]$ is a closed interval in $\mathbb{R}$ ($a < b$). By a *partition* of I we mean a finite collection of points $\{t_i\}_{i=0}^m \subset I$ for some $m \in \mathbb{N}$ such that $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$, which is written as $\{t_i\}_0^m \prec I$.