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Submanifolds of Cartan–Hartogs domains and complex Euclidean spaces

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ABSTRACT

We study the non-existence of common submanifolds of a complex Euclidean space and a Cartan–Hartogs domain equipped with their canonical metrics.

1. Introduction

The problem of holomorphic isometric embedding is a classical problem studied by many mathematicians. In a celebrated paper by Calabi [1], many deep results of a holomorphic isometry from a complex manifold into a complex space form have been obtained by using his diastasis functions. In particular, given two complex space forms with different curvature signs, Calabi proved that there does not exist local holomorphic isometric embedding between them with respect to the canonical Kähler metrics. Di Scala and Loi later generalized Calabi's non-embeddability result to Hermitian symmetric spaces of different types in [3].

Following Calabi's idea of diastasis, Umehara [9] studied the existence of common submanifolds of two Kähler manifolds and proved that two complex space forms with different curvature signs cannot have a common Kähler submanifold with the induced metrics. In [4], Di Scala and Loi called two Kähler manifolds are relatives when they share a common Kähler submanifold. They also proved that a bounded domain with its Bergman metric cannot be a relative to a projective algebraic manifold with the induced Fubini–Study

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metric. In fact, the result of Umehara [9] implies that the complex Euclidean space and a projective algebraic manifold with the induced Fubini–Study metric cannot be relatives. More recently, Huang and Yuan [6] proved that a complex Euclidean space and a Hermitian symmetric space of noncompact type cannot be relatives by using different argument. For related problems, the interested reader may refer to [2] and [12].

Cartan–Hartogs domains, introduced by Yin and Roos, are fiber bundles over classical domains in complex Euclidean spaces. They are natural generalizations of bounded symmetric domains and ellipsoids, but in general they are not homogeneous and their Bergman kernel functions are not rational. The Bergman kernels/metrics, proper holomorphic maps of Cartan–Hartogs domains have received considerable attentions thanks to the works by Loi–Zedda [7], Tu–Wang [8], Yin [10], etc. In [11], it was asked whether a complex Euclidean space and a Cartan–Hartogs domain can be relatives. We try to answer this question in this note.

Let the bounded symmetric domain Ω be the Harish-Chandra realization of an irreducible Hermitian symmetric space of noncompact type and let $N_\Omega(z,w)$ be its generic norm. The Cartan–Hartogs domain is defined as $M_\Omega(\mu)=\{(z,w)\in\Omega\times\mathbb{C}^N \,|\, |w|^{2\mu}<N_\Omega(z,z)\}$, where N is a positive integer, μ is a positive real number. In particular, when Ω is one of the classic domains of Type I, II, III, IV, Yin [10] obtained the Bergman kernels $K_{M_\Omega(\mu)}$ as follows:

- When Ω is of Type I,

$$K_{M_\Omega(\mu)}=K_I(z,w,\bar{z},\bar{w})=\mu^{-pq}\pi^{-(pq+N)}C(Y)det(I-z\bar{z}^T)^{-(p+q+\frac{N}{\mu})},$$

where $z\in\Omega$ is a $m\times n$ matrix, $C(Y)=\sum_{i=0}^{pq+1}b_{1i}\Gamma(N+i)Y^{N+i}$, $b_{1i}\in\mathbb{R}$, $Y=(1-X)^{-1}$, $X=|w|^2(det(I-z\bar{z}^T))^{\frac{-1}{\mu}}$;
- When Ω is of Type II,

$$K_{M_\Omega(\mu)}=K_{II}(z,w,\bar{z},\bar{w})=\mu^{\frac{-(p+1)}{2}}\pi^{-(\frac{p(p+1)}{2}+N)}C(Y)det(I-z\bar{z}^T)^{-(p+1+\frac{N}{\mu})},$$

where $z\in\Omega$ is a $p\times p$ skew-symmetric matrix, $C(Y)=\sum_{i=0}^{\frac{p(p+1)}{2}+1}b_{2i}\Gamma(N+i)Y^{N+i}$, $b_{2i}\in\mathbb{R}$, Y and X are the same as above;
- When Ω is of Type III,

$$K_{M_\Omega(\mu)}=K_{III}(z,w,\bar{z},\bar{w})=\mu^{\frac{-q(q-1)}{2}}\pi^{-(\frac{q(q-1)}{2}+N)}C(Y)det(I-z\bar{z}^T)^{-(q-1+\frac{N}{\mu})},$$

where $z\in\Omega$ is a $q\times q$ skew-symmetric matrix, $C(Y)=\sum_{i=0}^{\frac{q(q-1)}{2}+1}b_{3i}\Gamma(N+i)Y^{N+i}$, $b_{3i}\in\mathbb{R}$, Y and X are the same as above;
- When Ω is of Type IV,

$$K_{M_\Omega(\mu)}=K_{IV}(z,w,\bar{z},\bar{w})=\mu^{-n}\pi^{-(n+N)}C(Y)(1+|zz^T|^2-2z\bar{z}^T)^{-(n+\frac{N}{\mu})},$$

where $z\in\Omega\subset\mathbb{C}^n$, $C(Y)=\sum_{i=0}^{n+1}b_{4i}\Gamma(N+i)Y^{N+i}$, $b_{4i}\in\mathbb{R}$, Y is the same as above, $X=|w|^2(1+|zz^T|^2-2z\bar{z}^T)^{\frac{-1}{\mu}}$.

For convenience, we consider $M_\Omega(\mu)$ as a subset in $\mathbb{C}^{pq+N}$ in the following sense, where $pq=p^2$ when Ω is of type II; $pq=q^2$ when Ω is of type III; $pq=n$ when Ω is of type IV. The Bergman metric on $M_\Omega(\mu)$ is given by $\omega_{M_\Omega(\mu)}=\sqrt{-1}\partial\bar{\partial}\log K_{M_\Omega(\mu)}$ up to a positive normalization constant. Assume that $D\subset\mathbb{C}^\kappa$ is a connected open set and ω_D is a Kähler metric on D which is not necessarily complete. We assume, without loss of generality, 0 is contained in D. The question raised in [11] asks whether there simultaneously exist holomorphic isometric immersions $F:(D,\omega_D)\to(\mathbb{C}^n,\omega_{\mathbb{C}^n})$ and $L:(D,\omega_D)\to(M_\Omega(\mu),\omega_{M_\Omega(\mu)})$. In this note, we show that there do not exist such immersions mapping 0 to 0.