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Two open problems of Day and Wong on left thick subsets and left amenability

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This paper answers two open problems raised by Mahlon M. Day [4] and James C.S. Wong [17]: Does uniformly topological left amenability imply the existence of left translation continuous measures on a locally compact semitopological semigroup? Let T be a locally compact Borel subsemigroup of a locally compact semitopological semigroup S. Is the existence of a topological T-invariant mean M on S with $M(\chi_T) > 0$ enough to imply the topological left amenability of T? We give a negative answer to the first question by providing a counterexample and a positive proof to the second. This example can also be used to show that the property (α) provided in Gerard L. Sleijpen [14] can not be removed in one of his results.

1. Introduction

The study of left thick subsets goes back to Mitchell [13], in which he introduced this notion to characterize those subsets that support left invariant means of a discrete semigroup S. A characterization of left thick subsets of S without assuming left amenability is that a subset T is left thick if and only if the closure of T in βS, the Stone–Čech compactification of S, contains a minimal left ideal (see [6]). Left thick subsets have been used to calculate the cardinality of left invariant means of discrete semigroups and of locally compact groups (see [10,12]). Lau [11] used left thick subsets of a discrete semigroup S to construct extremely left amenable (ELA) subalgebras of bounded complex valued functions on S, and proved that every ELA subalgebra is contained in an algebra of such constructed. In [8] and [9], left thickness with respect to subalgebras of bounded complex valued functions on discrete semigroups was defined and generalized the theory further, including the important hereditary property for left thick discrete subsemigroups proved in [13]:

Theorem 1.1. *Let T be a left thick subsemigroup of a discrete semigroup S. Then T is left amenable if and only if S is left amenable.*

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This follows from a result of Day [2] which states:

Theorem 1.2. *Let S be a semigroup with left invariant mean ϕ. Suppose T is a subsemigroup of S with $\phi(\xi_T) > 0$, where ξ_T is the characteristic function of T on S, i.e., $\xi_T(s) = 1$ if $s \in T$, $\xi_T(s) = 0$ if $s \notin T$. Then T is also left amenable.*

For a locally compact semitopological semigroups S, we are interested in the measure algebra $M(S)$ and positive linear functionals on $M(S)^*$. Without assuming topological left amenability of S, Day [3] and Wong [15] introduced topological left lumpy (TLL) and topological left thick (TLT) subsets respectively as topological analogues of left thick subsets of discrete semigroups. When S is topological left amenable, both TLL and TLT subsets support topological left invariant means but support different types of nets in $P(S)$, the set of all probability measures on S.

Let μ_α be a net in $P(S)$, we call it:

a. (LSP), if $\|\nu * \mu_\alpha - u_\alpha\| \to 0$ for all $\nu \in P(S)$.
b. (LSU), if either of the following holds (see [4]):
   b_1. for each compact subset $K \subset S$, $\|\nu * \mu_\alpha - u_\alpha\| \to 0$ uniformly for all $\nu \in P(S)$ supported on K.
   b_2. for each compact subset $K \subset S$, $\|\delta_s * \mu_\alpha - u_\alpha\| \to 0$ uniformly for all $s \in K$.

Day showed in [4] that a (LSP) net is supported on a TLL subset while a (LSU) net is supported on a TLT subset. It is well-known that S is topological left amenable (TLA) if and only if it has a (LSP) net (see e.g. [4]). Correspondingly, we say S is uniformly topological left amenable (U-TLA) if it has a (LSU) net.

The equivalence of TLL and TLT was proved in [4, Theorem 5.6] under two conditions:

1) $L^n(S) := \{\mu \in M(S) \mid s \to \delta_s * |\mu| \text{ is continuous}\}$ is nontrivial, i.e., $L^n(S) \neq \{0\}$.
2) S is U-TLA.

We know that both conditions hold when S is a left amenable locally compact group. It is also known that when S is TLA, 1) implies 2) (see e.g. [7]). So Day asked if 2) implies 1) (see [4, p. 84]). We shall answer his question in the negative in Example 2.1. We shall also show with Example 2.1 that property (α) in [14, Proposition 4.2] can not be removed.

In Day [4], a generalization of Theorem 1.1 was given as follows:

Theorem 1.3. *Let T be a locally compact TLL subsemigroup of a locally compact semitopological semigroup S. Then T is TLA if and only if S is TLA.*

However, as Wong pointed out in [17], the generalization of Theorem 1.2 is an open problem. We shall give a positive answer to this problem in Theorem 3.2.

2. Definitions and results related to thick subsets

A semitopological semigroup is a tripe (S, ρ, τ) where S is a set, ρ is an associative binary operation on S and τ is a topology on S such that ρ is separately continuous. Let $s, t \in S$, we write $\rho(st) = st$ for short. For general theory of semitopological semigroup, we refer to [1]. Throughout this paper S is a locally compact semitopological semigroup; $M(S)$ is the Banach space of regular complex Borel measures on S with total variation norm, and $P(S)$ is the set of probability measures on S. For each subset E of S, let χ_E be the characteristic functional of E on $M(S)$. We write $\mathbb{1}$ for χ_S. A positive linear functional M on $M(S)^*$ is said to be a mean if $\|M\| = M(\mathbb{1}) = 1$. We say a Borel subset T of S supports a mean M if $M(\chi_T) = 1$. We