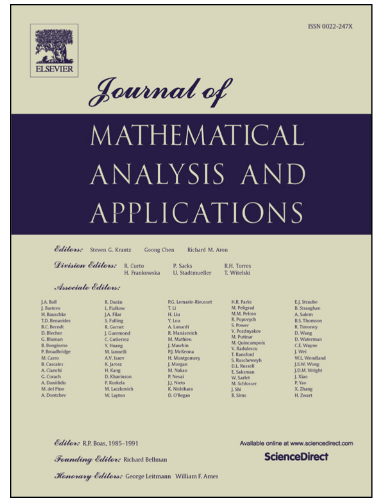

Smooth compositions with a nonsmooth inner function

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SMOOTH COMPOSITIONS WITH A NONSMOOTH INNER FUNCTION

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Abstract. Let $p : \mathbb{R} \to \mathbb{R}$ be a given function, and let $\mathcal{A}_p$ be the set of smooth functions f such that $f(p(\cdot) + c)$ is smooth for any $c \in \mathbb{R}$. We show that if p is not smooth, then either every element of $\mathcal{A}_p$ is constant, or there is a nonzero constant d such that $\mathcal{A}_p$ equals to the set of smooth functions of periodicity d.

1. Introduction

In this paper, we prove the following result:

Theorem 1.1. *Let n be a nonnegative integer or ∞, $p : \mathbb{R} \to \mathbb{R}$ be a given function with $p \notin C^n(\mathbb{R}, \mathbb{R})$ and*

$$\mathcal{A}_p = \{f \in C^n(\mathbb{R}, \mathbb{R}) \mid f(p(\cdot) + c) \in C^n(\mathbb{R}, \mathbb{R}) \text{ for any } c \in \mathbb{R}\}. \tag{1.1}$$

Then, either $\mathcal{A}_p = \mathbb{R}$ or $\mathcal{A}_p = C^n_d(\mathbb{R}, \mathbb{R})$ for some nonzero constant d. Here, $\mathcal{A}_p = \mathbb{R}$ means that every element in $\mathcal{A}_p$ is constant, and $C^n_d(\mathbb{R}, \mathbb{R})$ means the collection of all functions in $C^n(\mathbb{R}, \mathbb{R})$ of periodicity d.

This result is motivated by the work [1] of Christensen and Wu on diffeological vector spaces. A weaker form of Theorem 1.1 is needed in [1]. The method in [1] dealing with the weaker form seems not applicable to prove Theorem 1.1.

It is clear that $\mathcal{A}_p$ is a translation invariant subalgebra of $C^n(\mathbb{R}, \mathbb{R})$. As an example, take $p = \chi_E$ with E an abitrary subset of $\mathbb{R}$ ($E \neq \emptyset, \mathbb{R}$). Then, it is clear that $\cos(2\pi x), \sin(2\pi x) \in \mathcal{A}_p$. In fact, it is not hard to see that $\mathcal{A}_p = C^n_1(\mathbb{R}, \mathbb{R})$ in this specific example.

A key ingredient in the proof of Theorem 1.1 is the following result about the continuity of maps with σ-compact graph.

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