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Generalized Post–Widder inversion formula with application to statistics

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ABSTRACT

In this work we derive an inversion formula for the Laplace transform of a density observed on a curve in the complex domain, which generalizes the well known Post–Widder formula. We establish convergence of our inversion method and derive the corresponding convergence rates for the case of a Laplace transform of a smooth density. As an application we consider the problem of statistical inference for variance-mean mixture models. We construct a nonparametric estimator for the mixing density based on the generalized Post–Widder formula, derive bounds for its root mean square error and give a brief numerical example.

1. Introduction

Let p be a probability density on $\mathbb{R}_+$, then the integral

$$\mathcal{L}(z) := \int_0^\infty e^{-zx} p(x)\, dx, \quad \operatorname{Re} z > 0, \tag{1}$$

exists and is called the Laplace transform of p. The Laplace transform is a popular tool for solving differential equations and convolution integral equations. Its inversion is of importance in many problems from e.g. physics, engineering and finance (cf. [2] and [6] for various examples).

In general, the complexity of the inversion problem for $\mathcal{L}$ depends on the information available about the Laplace transform. If the Laplace transform is explicitly given on its half-plane of convergence, the density p can be reconstructed using the so-called Bromvich contour integral (see, e.g. [7])

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$$p(x) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{zx}\mathcal{L}(z)\,dz, \quad x > 0.$$

In the real case, i.e. in the situation where the Laplace transform of p is known on the real axis only, the inversion of $\mathcal{L}$ is a well-known ill-posed problem (see for example [4,5] and references therein). One popular solution for this case is given by the well known Post–Widder formula, which reads as follows (cf. [7]):

$$p(x) = \lim_{N\to\infty} \frac{(-1)^N}{N!}\left(\frac{N}{x}\right)^{N+1} \mathcal{L}^{(N)}\left(\frac{N}{x}\right).$$

In some situations, the Laplace transform $\mathcal{L}$ can only be computed on some curve ℓ in $\mathbb{C}$, which is different from $\mathbb{R}_+$ or $\{\mathrm{Re}(z) = c\}$ for some $c > 0$. In this paper we generalize the Post–Widder formula to the case of rather general curves ℓ and derive the convergence rates of the resulting estimator.

As an application of our results we consider the problem of estimating the mixing density in a variance-mean mixture model (see e.g. [1] and [3]). After constructing the estimator, we derive bounds for its root mean square error (RMSE) and demonstrate its performance in a short numerical example. An advantage of using the generalized Post–Widder formula here is that the resulting estimator can be evaluated without any numerical integration.

The paper is organized as follows. In Section 2 we introduce the generalized Post–Widder inversion formula and discuss its convergence behavior. Section 3 is devoted to the statistical inference for variance-mean mixtures together with some numerical results. Finally, the proofs of our results are given in Section 4 to 6.

2. Generalized Post–Widder Laplace inversion

In this section we will introduce a generalized Post–Widder inversion formula that extends the classical result by Post and Widder [7] to the situation when the Laplace transform of a continuous density on $[0,\infty]$ is given on a curve in the complex plane. Subsequently, we prove a convergence result and derive the rates of convergence for the resulting inverse Laplace transform.

2.1. Inversion formula and its kernel representation

Let p be a continuous probability density on $[0,\infty)$ and let its Laplace transform $\mathcal{L}(z)$ be given on a curve:

$$\ell := \{z = y + \mathrm{i}c(y) : y \in \mathbb{R}_+\}, \tag{2}$$

such that c is piecewise smooth with $c(y) = o(y)$ as $y \to \infty$. In this setting the generalized Post–Widder formula can be described as follows.

Definition 2.1 *(Generalized Post–Widder formula).* For any fixed $x > 0$, we introduce the generalized Post–Widder formula by

$$p_N(x) := \frac{(-1)^N}{N!}\left(g\left(\frac{N}{x}\right)\right)^{N+1} \mathcal{L}^{(N)}\left(g\left(\frac{N}{x}\right)\right), \tag{3}$$

where $\mathcal{L}^{(N)}$ denotes the Nth-derivative of the Laplace transform $\mathcal{L}$ and $g(y) := y + \mathrm{i}c(y)$. For fixed $x > 0$, we define the generalized Post–Widder kernel via

$$K_N(t,x) := \frac{\left(N + \mathrm{i}xc\left(\frac{N}{x}\right)\right)^{N+1}}{N!} t^N e^{-(N+\mathrm{i}c(\frac{N}{x})x)t}, \quad t > 0. \tag{4}$$