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Unconditionally convergent multipliers and Bessel sequences ☆

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We prove that every unconditionally summable sequence in a Hilbert space can be factorized as the product of a square summable scalar sequence and a Bessel sequence. Some consequences on the representation of unconditionally convergent multipliers are obtained, thus providing positive answers to a conjecture by Balazs and Stoeva in some particular cases.

1. Introduction

A *multiplier* on a separable Hilbert space H is a bounded operator

$$M_{m,\Phi,\Psi} : H \to H,\ f \mapsto \sum_{n=1}^{\infty} m_n \langle f, \Psi_n \rangle \Phi_n,$$

where $\Phi = (\Phi_n)_n$ and $\Psi = (\Psi_n)_n$ are sequences in H and $m = (m_n)_n$ is a scalar sequence called the symbol. These operators are generalizations of Gabor multipliers, which in turn are discrete versions of time-frequency localization operators introduced by Daubechies [7]. They have found applications in the analysis of pseudo-differential operators [6] and multi-window spectrograms [4,1], which can be used for spectral estimation. Due to its discrete nature, multipliers are more akin to the implementations required in acoustics [3].

The multiplier is said to be unconditionally convergent if the above series converges unconditionally for every $f \in H$. For any (unconditionally convergent) multiplier $M_{m,\Phi,\Psi}$ its adjoint $M_{\overline{m},\Psi,\Phi}$ is also a (unconditionally convergent) multiplier.

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Observe that each bounded operator T on H can be expressed as a multiplier: if $(u_n)_n$ is an orthonormal basis, we can take $\Phi_n = Tu_n$, $\Psi_n = u_n$ (alternatively $\Phi_n = u_n$, $\Psi_n = T^*u_n$) and $m_n = 1$ for each $n \in \mathbb{N}$.

In the case that $\Phi = (\Phi_n)_n$ and $\Psi = (\Psi_n)_n$ are Bessel sequences in H and $m \in \ell^\infty$ the operator $M_{m,\Phi,\Psi}$ is called a Bessel multiplier. Recall that $\Psi = (\Psi_n)_n$ is called a *Bessel sequence* if there is a constant $B > 0$ such that

$$\sum_{n=1}^{\infty} |\langle f, \Psi_n\rangle|^2 \leq B\|f\|^2$$

for every $f \in H$. It turns out that $(\Psi_n)_n$ is a Bessel sequence if and only if there exists a bounded operator $T : \ell^2 \to H$ such that $T(e_n) = \Psi_n$, where $(e_n)_n$ denote the canonical unit vectors of ℓ^2 ([5, Theorem 3.2.3]).

Bessel multipliers were introduced and studied in a systematic way by Balazs [2] as a generalization of the Gabor multipliers considered in [9]. In [2] it is proved that each Bessel multiplier is unconditionally convergent. Balazs and Stoeva [14] provide examples of non-Bessel sequences and non-bounded symbols defining unconditionally convergent multipliers. However all the examples are obtained from a Bessel multiplier after some trick. In fact, Balazs and Stoeva *conjecture* in [13] that every unconditionally convergent multiplier can be written as a Bessel multiplier with constant symbol by shifting weights. More precisely, if $M_{m,\Phi,\Psi} : H \to H$ is an unconditionally convergent multiplier, then they conjecture that there exist scalar sequences $(a_n)_n,\ (b_n)_n$ such that

$$m_n = a_n \cdot \bar{b}_n$$

and

$$(a_n\Phi_n)_n\,,\ (b_n\Psi_n)_n$$

are Bessel sequences in H. Several classes of multipliers for which the conjecture is true are obtained in [13]. For instance, they proved that this is the case for multipliers of the form $M_{m,\Phi,\Phi}$ [13, Proposition 4.2] and also for multipliers with the property that the sequence $(|m_n|\,||\Phi_n||\,||\Psi_n||)_n$ is norm bounded below [13, Proposition 1.1].

In the particular case that $m_n = 1$ and $\Psi_n = g$ for every $n \in \mathbb{N}$, the conjecture has a positive answer if and only if for every unconditionally summable sequence $(\Phi_n)_n$ in a separable Hilbert space H we may find $(\alpha_n)_n \in \ell^2$ such that $(\frac{1}{\alpha_n}\Phi_n)_n$ is a Bessel sequence in H. So, the main aim of the present paper is to analyze the structure of unconditionally summable sequences in a separable Hilbert space. As a consequence we obtain some new situations where the conjecture of Balazs and Stoeva is still true, which are different in spirit to the ones considered in [13]. Our results cannot be considered as improvements of those in [13] nor can be obtained with the same techniques, they cover a completely different situation as in the cases we consider the sequence $(|m_n|\,||\Phi_n||\,||\Psi_n||)_n$ converges to zero.

2. Results

We will use the well known fact that a series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is unconditionally convergent if and only if there exist a compact operator $T : c_0 \to X$ with the property that $T(e_n) = x_n$, where $(e_n)_n$ denote the canonical unit vectors of c_0 (see for instance the omnibus theorem on unconditional summability in [8, 1.9]). We recall that, in the case that $X = H$ is a Hilbert space, every bounded operator $T : c_0 \to H$ is compact. In fact, the closed unit ball B of H is weakly compact, the transposed map $T^* : H \to \ell^1$ is a bounded operator and weak and norm convergence of sequences in ℓ^1 coincide ([8, Theorem 1.7]). Therefore T^* is a compact operator and so is T.