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Multiple recurrence theorems for set-valued maps

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ABSTRACT

In this paper, we mainly extend the Khintchine's recurrence theorem, Furstenberg's multiple recurrence theorem, and multiple Birkhoff recurrence theorem from single-valued maps to set-valued maps.

1. Introduction

Both for its own sake and for application to the usual function case, set-valued dynamical systems have recently received some attention, see e.g., J.-P. Aubin, H. Frankowska and A. Lasota (1991 [2,3]), W. Miller and E. Akin (1999 [11]), and X. Dai (2017 [6]).

Recurrence plays an important role in dynamics and has many applications to combinatorics and number theory. Specifically, when the underlying space of the dynamical system is appropriately bounded in the sense of topology or measure theory, some orbits will necessarily exhibit some form of recurrence, in that they return close to their original position. The first result of this type (for a measure preserving transformation on a finite measure space) was formulated by H. Poincaré [12]. The second result of this kind (for a continuous transformation on a compact metric space) was due to G.D. Birkhoff [5]. In [10], A.Y. Khintchine strengthened the Poincaré recurrence theorem, showing that: If T is a measure preserving transformation of a probability space $(X, \mathcal{B}, \mu)$, and $A \in \mathcal{B}$ with $\mu(A) > 0$, then for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} \mid \mu(A \cap T^{-n}(A)) > \mu(A)^2 - \varepsilon\}$ is syndetic.

For a set-valued dynamical system, the Poincaré's recurrence theorem has been studied by the authors of [3]. They first introduced the notion of an invariant measure of a set-valued map, and proved the existence theorem of an invariant measure: Let X be a compact metric space and $F : X \rightsquigarrow X$ be a closed set-valued

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map with nonempty values (a set-valued map F is called *closed* if its graph $:= \{(x, y) \in X \times X \mid y \in F(x)\}$ is closed in $X \times X$), then there exists an invariant probability measure μ, i.e., satisfying $\forall A \in \mathcal{B}(X), \mu(A) \leq \mu(F^{-1}(A))$. Finally they proved the Poincaré recurrence theorem: Let X be a compact metric space, $F : X \rightsquigarrow X$ be a closed set-valued map and $\mu \in \mathcal{P}(X)$ an invariant measure of F. For any Borel subset $B \subset X$, let

$$B_\infty = \cap_{N \geq 0} \cup_{n \geq N} F^{-n}(B)$$

be the subset of points x such that for all N, there exists $n \geq N$ such that $F^n(x) \cap B \neq \emptyset$. Then the measure of $B \cap B_\infty$ is equal to the measure of B.

More recently, the authors of [1] and [11] introduced several concepts which extend this idea of invariance for a measure, and discussed when these concepts are equivalent. In [11], Miller and Akin first introduced the idea of chain recurrence for the set-valued system and discussed the chain relations, finally gave a similarly close relationship between chain recurrence for the set-valued system and for the shift on the sample path space.

In 1977, H. Furstenberg extraordinarily extended the Poincaré recurrence theorem to *multiple recurrence* (see [7,8]):

Theorem 1.1. *If $T_1, \cdots, T_l$ are commuting measure preserving transformations of a probability space $(X, \mathcal{B}, \mu)$, and $A \in \mathcal{B}$ with $\mu(A) > 0$, then*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T_1^{-n}(A) \cap \cdots \cap T_l^{-n}(A)) > 0.$$

And by using ergodic theory he applied this result to prove the famous Szemerédi's theorem. Moreover, he and Y. Katznelson [9] strengthened this result, showing that under the same assumptions: $\{n \in \mathbb{N} \mid \mu(A \cap T_1^{-n}(A) \cap \cdots \cap T_l^{-n}(A)) > 0\}$ *is an IP*-set.*

In 1978 [8], Furstenberg and B. Weiss proved a topological analogue – the multiple Birkhoff recurrence theorem:

Theorem 1.2. *Let $T_1, \cdots, T_l$ be commuting continuous transformations of a compact metric space X. Then there exists a point $x \in X$, and a sequence $n_k \to \infty$, with $T_i^{n_k}(x) \to x$ simultaneously for $i = 1, \cdots, l$.*

These results have been applied to combinatorics and number theory.

In 2017 [6], X. Dai first considered a commuting family of upper semicontinuous set-valued maps with closed nonempty values, and proved that invariant measures for this case exist by using a multiple univalent lifting technique.

In this paper, we are only interested in the multiple recurrence theorems for commuting set-valued maps.

First we shall introduce some "lifting" technique that describes a relation between multiple recurrence for set-valued maps and single-valued maps (see Lemma 3.2). Then based on the "lifting" technique, we shall prove that for measure preserving set-valued maps, the classical Khintchine's recurrence theorem and Furstenberg's multiple recurrence theorem still hold on some special "invariant" subsets with positive measure (see Theorems 3.1–3.3).

Finally, using a multiple univalent lifting technique motivated by [6], we extend the classical multiple Birkhoff recurrence theorem from continuous transformations to upper semicontinuous set-valued maps with closed nonempty values (see Theorem 3.4). Moreover we provide a simple application to two commuting single-valued maps and obtain a new phenomenon of recurrence for this case (see Corollary 3.3).

Throughout this paper, undefined terminology should be referred to [8].