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The variance conjecture on projections of the cube ☆

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ABSTRACT

We prove that the uniform probability measure μ on every $(n-k)$-dimensional projection of the n-dimensional unit cube verifies the variance conjecture with an absolute constant C

$$\mathrm{Var}_\mu |x|^2 \leq C \sup_{\theta\in S^{n-1}} \mathbb{E}_\mu \langle x,\theta\rangle^2 \mathbb{E}_\mu |x|^2,$$

provided that $1 \leq k \leq \sqrt{n}$. We also prove that if $1 \leq k \leq n^{\frac{2}{3}}(\log n)^{-\frac{1}{3}}$, the conjecture is true for the family of uniform probabilities on its projections on random $(n-k)$-dimensional subspaces.

1. Introduction and notation

The (generalized) variance conjecture states that there exists an absolute constant C such that for every centered log-concave probability μ on $\mathbb{R}^n$ (i.e. of the form $d\mu = e^{-v(x)}dx$ for some convex function $v : \mathbb{R}^n \to (-\infty, \infty]$)

$$\mathrm{Var}_\mu |x|^2 \leq C\lambda_\mu^2 \mathbb{E}_\mu |x|^2,$$

where $\mathbb{E}_\mu$ and Var_μ denote the expectation and the variance with respect to μ and λ_μ is the largest eigenvalue of the covariance matrix, i.e. $\lambda_\mu^2 = \max_{\theta\in S^{n-1}} \mathbb{E}_\mu\langle x,\theta\rangle^2$ where S^{n-1} denotes the unit Euclidean sphere in $\mathbb{R}^n$.

This conjecture was first considered in the context of the so called Central Limit Problem for isotropic convex bodies in [7] and it is a particular case of a more general statement, known as the Kannan, Lovász, and Simonovits or KLS-conjecture, see [9], which conjectures the existence of an absolute constant C such

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that for any centered log-concave probability in $\mathbb{R}^n$ and any locally Lipschitz function $g:\mathbb{R}^n\to\mathbb{R}$ such that $\mathrm{Var}_\mu g(x)$ is finite

$$\mathrm{Var}_\mu\, g(x)\le C\lambda_\mu^2\mathbb{E}_\mu\,|\nabla g(x)|^2.$$

In recent years a number of families of measures have been proved to verify these conjectures (see [2] for a recent review on the subject). For instance, the KLS-conjecture is known to be true for the Gaussian probability and the uniform probability measures on the ℓ_p^n-balls, some revolution bodies, the simplex and, with an extra $\log n$ factor, on unconditional bodies and log-concave probabilities with many symmetries (see [4–6,8,10,11,14]). The best general known result for the KLS-conjecture adds a factor $\sqrt{n}$ and is due to Lee and Vempala (see [12]). Besides, the variance conjecture is known to be true for uniform probabilities on unconditional bodies and on hyperplane projections of the cross-polytope and the cube (see [10] and [1]). The best general estimate for the variance conjecture is the one given by Lee and Vempala for the KLS-conjecture.

We would like to remark that, while in the case of the KLS-conjecture one can assume without loss of generality that μ is isotropic (since then every linear transformation of the measure verifies it) this is not the case when we restrict to the variance conjecture, as we are considering only the function $g(x)=|x|^2$.

Before stating our results let us introduce some more notation. Let

$$B_\infty^n:=\{x\in\mathbb{R}^n\,:\,|x_i|\le 1,\,\forall\,1\le i\le n\}$$

denote the n-dimensional unit cube and, for any $1\le k\le n$, let $G_{n,n-k}$ be the set of all $(n-k)$-dimensional subspaces of $\mathbb{R}^n$. For any $E\in G_{n,n-k}$ we will denote by $K:=P_EB_\infty^n$ the orthogonal projection of B_∞^n onto E and by μ the uniform probability on K. $\{e_i\}_{i=1}^n$ will denote the standard canonical basis in $\mathbb{R}^n$. As mentioned before, it was proved in [1] that the family of uniform probabilities on any $(n-1)$-dimensional projection of B_∞^n verifies the variance conjecture.

In this paper we will prove the following

Theorem 1.1. *There exists an absolute constant C such that for any $1\le k\le\sqrt{n}$ and any $E\in G_{n,n-k}$, if μ denotes the uniform probability measure on $K=P_EB_\infty^n$, then*

$$Var_\mu|x|^2\le C\lambda_\mu^2\mathbb{E}_\mu|x|^2.$$

We will also prove the following theorem, which shows that for k in a larger range, the variance conjecture is true for the family of uniform probabilities on the projections of B_∞^n on a random $(n-k)$-dimensional subspace. For that matter, we denote by $\mu_{n,n-k}$ the Haar probability measure on $G_{n,n-k}$.

Theorem 1.2. *There exist absolute constants C,c_1,c_2 such that for any $1\le k\le\frac{n^{\frac{2}{3}}}{(\log n)^{\frac{1}{3}}}$, if μ denotes the uniform probability measure on $K=P_E(B_\infty^n)$, the measure $\mu_{n,n-k}$ of the subspaces $E\in G_{n,n-k}$ for which*

$$Var_\mu|x|^2\le C\lambda_\mu^2\mathbb{E}_\mu|x|^2$$

is greater than $1-c_1e^{-c_2n^{\frac{2}{3}}(\log n)^{\frac{2}{3}}}$.

The main tool to prove both theorems will be to decompose an integral on K as the sum of the integrals on the projections of some $(n-k)$-dimensional faces. It was proved in [3] that for any $E\in G_{n,n-k}$ there exist $F_1,\dots,F_l$ a set of $(n-k)$-dimensional faces of B_∞^n such that for any integrable function f on K

$$\mathbb{E}_\mu f:=\frac{1}{|K|}\int\limits_K f(x)dx=\sum_{i=1}^l\frac{|P_E(F_i)|}{|K|}\mathbb{E}_{P_E(F_i)}f(x)$$