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Spherical arc-length as a global holomorphic parameter for analytic curves in the Riemann sphere ☆

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ABSTRACT

We prove that for every analytic curve in the complex plane $\mathbb{C}$, Euclidean and spherical arc-lengths are global holomorphic parameters. We also prove that for any analytic curve in the hyperbolic plane, hyperbolic arc-length is also a global parameter. We generalize some of these results to the case of analytic curves in $\mathbb{R}^n$ and $\mathbb{C}^n$ and we discuss the situation of curves in the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

1. Introduction

The subject of this paper is the interaction between real analytic curves and complex geometry. The results are in the direction of those obtained in the papers [4,5]. Before stating our results, we recall a few notions from these two papers in order to make the present paper self-contained.

Definition 1.1. Let $I \subset \mathbb{R}$ be a nonempty interval of an arbitrary type which is not a singleton and let $\vec{\gamma} : I \to \mathbb{K}^m$ be a continuous map, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We say that $\vec{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ is a *(regular) analytic curve* if $\vec{\gamma}$ is differentiable with non-zero derivative at each point, and such that for every $t_0 \in I$ there exists $\delta = \delta_{t_0} > 0$ and a power series $\sum_{n=0}^{\infty} (t - t_0)^n \vec{a}_n$, $\vec{a}_n \in \mathbb{K}^m$, which is convergent on $(t_0 - \delta, t_0 + \delta)$

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and such that $\vec{\gamma}(t) = \sum_{n=0}^{\infty}(t - t_0)^n \vec{a}_n$ for all $t \in I \cap (t_0 - \delta, t_0 + \delta)$. The word "regular" is to signify that the derivative is non-zero at each point and is usually omitted. If a curve parametrized by t satisfies this definition we say that the parameter t is a holomorphic parameter for the curve and that the curve is a holomorphic mapping.

If one can further extend an analytic curve $\vec{\gamma} : I \to \mathbb{K}^n$ to an analytic curve defined on a larger interval than I, we call this extension an analytic extension of $\vec{\gamma}$. Every analytic curve has a maximal analytic extension. Here, the word *maximal* refers to set-theoretic inclusion between domains of definition. Note however that a maximal extension depends on the parametrization of a curve and not only on its image. It was shown in [4,5] that if we re-parametrize an analytic curve by arc-length, then the resulting curve is still analytic, and that the maximal analytic extension with respect to arc-length is a canonical maximal extension in the sense that its domain of definition contains the domain of definition of the original analytic curve (and, hence, of any analytic re-parametrization of the curve).

A holomorphic parameter for an analytic curve is called *global* if it is a holomorphic parameter for the maximal analytic extension of the curve. Hence a global holomorphic parameter is the restriction of a holomorphic mapping, but need not be globally injective. In the paper [5], it was also proved that arc length is a *global* holomorphic parameter for every analytic curve.

In the present paper we show the existence of other global holomorphic parameters of arbitrary analytic curves. For instance spherical arc-length is such a global holomorphic parameter. We also introduce and investigate several other kinds of lengths.

In §2 we begin with the classical case of analytic curves in $\mathbb{C}$.

In §3, we extend the result of §2 to analytic curves in $\mathbb{R}^n$ and $\mathbb{C}^n$. Essentially, the same methods of proof work.

In §4, we consider the notion of analytic curve in the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The more general form of this definition is that locally the function γ is the restriction of an injective meromorphic function. We show that the spherical arc-length of such a curve is a global holomorphic parameter. It follows that the maximal extension of such a curve parametrized by spherical arc-length is always defined on an open subinterval of $\mathbb{R} = (-\infty, +\infty)$ and cannot contain the point $+\infty$. This contrasts with what happens if we use other parametrizations, for instance the Euclidean arc-length. We note that these parametrizations by Euclidean arc-length or spherical arc-length should define strictly increasing functions, and in most cases we are led to allow the parameter to take negative values, although we call it arc-length parameter or spherical arc-length parameter (see [5], beginning of §3, where this question is discussed).

Finally, we mention that our definitions are special cases of more general definitions of analytic curves on Riemann surfaces, but we do not insist on that in the present paper.

In this paper, by a *holomorphic* mapping we mean a mapping that is holomorphic with nonzero derivative, [1].

2. Analytic curves in $\mathbb{C}$

Let $\gamma : I \to \mathbb{C}$ be an analytic curve. It follows from the definition that γ is locally injective and that its maximal analytic extension is defined on an open subinterval of $\mathbb{R}$, see [5]. It is also true that if we restrict γ to a compact interval I_0 such that $\gamma_{|I_0}$ is injective, then γ has an injective holomorphic extension to an open neighborhood $V \subset \mathbb{C}$ of I_0 (see [4,5]). In fact, this local extendability to an injective holomorphic function is equivalent to the definition of an analytic curve [1–3].

Let $\gamma : [a, b] \to \mathbb{C}$ be an analytic curve and $N \geq 1$ a natural number. We consider the image in $\mathbb{C}^{2N}$ of $[a, b]$ by the map

$$\Gamma(t) = \left(\gamma(t), \overline{\gamma(\bar{t})}, \gamma'(t), \overline{\gamma'(\bar{t})}, \ldots, \gamma^{(N)}(t), \overline{\gamma^{(N)}(\bar{t})}\right).$$