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Some properties of the divided difference of psi and polygamma functions ☆

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ABSTRACT

Let $\psi_n = (-1)^{n-1}\psi^{(n)}$ for $n \geq 0$, where $\psi^{(n)}$ stands for the psi and polygamma functions. For $p, q \in \mathbb{R}$ and $\rho = \min(p, q)$, let

$$D[x+p, x+q; \psi_{n-1}] \equiv -\phi_n(x)$$

be the divided difference of the functions ψ_{n-1} for $x > -\rho$. In this paper, we establish the necessary and sufficient conditions for the function

$$\Phi_n(x, \lambda) = \phi_{n+1}(x)^2 - \lambda\phi_n(x)\phi_{n+2}(x)$$

to be completely monotonic on $(-\rho, \infty)$. In particular, we find that the function $\psi_{n+1}^2/(\psi_n\psi_{n+2})$ is strictly decreasing from $(0, \infty)$ onto $(n/(n+1),\ (n+1)/(n+2))$. These not only generalize and strengthen some known results, but also yield many new and interesting ones.

1. Introduction

The Euler's gamma and psi (digamma) functions are defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. The derivatives $\psi^{(n)}(x)$ for $n \in \mathbb{N}$ are called polygamma functions. We have the following integral and series representations [1, Section 6.3, 6.4]

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$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{k=1}^\infty \frac{1}{k(x+k)}, \tag{1.1}$$

$$(-1)^{n-1} \psi^{(n)}(x) = \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt} dt = \frac{n!}{x^{n+1}} + n! \sum_{k=1}^\infty \frac{1}{(x+k)^{n+1}}, \tag{1.2}$$

where $\gamma = 0.57721...$ denotes Euler's constant.

Over the decades, many scholars published remarkable properties of these functions. In particular, some interesting properties including monotonicity, convexity and inequalities for psi and polygamma functions can be found in review papers [28,36,16] and recent papers [10,13,14,29,20,44–46,48,37,39,47] and references therein.

It is well-known that many of inequalities for psi and polygamma functions are often consequences of complete monotonicity of these special functions. We recall that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n (f(x))^{(n)} \geq 0$ for $x \in I$ and $n \geq 0$ (see [8,41]). If this inequality is strict for all $x \in I$ and $n \geq 0$, then f is said to be strictly completely monotonic on I. The celebrated Bernstein–Widder's Theorem [41, p. 161, Theorem 12b] states that a function f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is nondecreasing and the integral converges for $0 < x < \infty$.

There exists a very extensive literature on completely monotonic functions related to gamma, psi and polygamma functions, see, e.g. [23,17,9,7,2,3,6,30,21,11,25,26,31,32,19,18,27,33,35,12,15,34,43,42].

For convenience, we denote by $\psi_0 = -\psi$ and $\psi_n = (-1)^{n-1} \psi^{(n)} = \left|\psi^{(n)}\right|$, $n \in \mathbb{N}$. Then $\psi_n' = -\psi_{n+1} < 0$, and so ψ_n for $n \in \mathbb{N}$ is strictly completely monotonic on $(0, \infty)$. In [6, Corollary 2.3], Alzer and Wells proved for $n \geq 2$, the function

$$F_n(x, c) = \psi_n(x)^2 - c\psi_{n-1}(x) \psi_{n+1}(x) \tag{1.3}$$

is strictly completely monotonic on $(0, \infty)$ if and only if $c \leq (n-1)/n$, and so is $-F_n(x, c)$ if and only if $c \geq n/(n+1)$. As a consequence, they got the following nice sharp inequalities

$$\frac{n-1}{n} < \frac{\psi_n(x)^2}{\psi_{n-1}(x) \psi_{n+1}(x)} < \frac{n}{n+1} \tag{1.4}$$

for $x > 0$ and $n \geq 2$.

Let us consider the divided difference of psi and polygamma functions $\psi^{(n-1)}$ for $n \in \mathbb{N}$ defined on $(-\min(p, q), \infty)$ by

$$\phi_n(x) = \begin{cases} (-1)^{n-1} \dfrac{\psi^{(n-1)}(x+p) - \psi^{(n-1)}(x+q)}{p-q} & \text{if } p \neq q, \\ (-1)^{n-1} \psi^{(n)}(x+q) & \text{if } p = q, \end{cases} \tag{1.5}$$

or equivalently,

$$\phi_n(x) = \frac{\int_q^p \psi_n(x+t) dt}{p-q} \text{ if } p \neq q \text{ and } \phi_n(x) = \psi_n(x+q). \tag{1.6}$$