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Circularly invariant uniformizable probability measures for linear transformations

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A B S T R A C T

In this paper, we prove a threshold result on the existence of a circularly invariant uniformizable probability measure (CIUPM) for linear transformations with non-zero slope on the line. We show that there is a threshold constant c depending only on the slope of the linear transformation such that there exists a CIUPM if and only if its support has a diameter at least as large as c. Moreover, the CIUPM is unique up to translation if the diameter of the support equals c.

1. Introduction

This paper investigates a variant of invariant probability measure for linear transformations on the line. Let $\mathbb{R}$ be the set of real numbers and $\mathbb{T}$ the unit circle identified with $[0, 1[$ via the canonical mapping $t \mapsto e^{2\pi i t}$, where the usual notations for intervals, e.g., $[a, b[= \{x \in \mathbb{R} : a \le x < b\}$ are used throughout. Let λ and $\lambda_{\mathbb{T}}$ be the Lebesgue measure on $\mathbb{R}$ and $\mathbb{T}$, respectively. Denote by $R : \begin{cases} \mathbb{R} \to \mathbb{T} \\ x \mapsto \langle x \rangle \end{cases}$ the rotation mapping with $\langle x \rangle$ being the fractional part of x, and $\operatorname{diam}(A) := \sup_{x,y \in A} |x - y|$ the diameter of a set $A \subset \mathbb{R}$. Note that $\operatorname{diam}(A) \ge \lambda(A)$ for every set A and the equality holds if and only if $\lambda(A) = +\infty$ or A is an interval but a Lebesgue measure zero set (i.e., $\lambda([\inf A, \sup A] \setminus A) = 0$ with $\inf A$ and $\sup A$ denoting the infimum and supremum of A, respectively). For every continuous monotone transformation $T : \mathbb{R} \to \mathbb{R}$, let $\mu \circ T^{-1}$ be the *induced* (or *push-forward*) probability measure for T, and $\langle \mu \rangle$ denotes $\mu \circ R^{-1}$ for convenience. Note

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that $\langle\mu\rangle$ and $\langle\mu\circ T^{-1}\rangle$ both are probability measures on $\mathbb{T}$. A measure μ on $\mathbb{R}$ is a *a circularly invariant uniformizable probability measure* (CIUPM) if

$$\langle\mu\rangle=\left\langle\mu\circ T^{-1}\right\rangle=\lambda_{\mathbb{T}}.$$

Obviously, every CIUPM is absolutely continuous (w.r.t. λ).

Our motivation for the study of CIUPMs comes from *uniform distribution theory*. For every sequence (x_n) of real numbers, let

$$\mu_n:=\mu_n(x_n)=\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}$$

be the sequence of probability measures on $\mathbb{R}$ generated by (x_n). It is known that for some convex monotone transformations $T:\mathbb{R}\to\mathbb{R}$ like the exponential transformation or the trivially convex linear transformation, there exists a *uniformly distributed modulo one* (u.d. mod 1) sequence (x_n) of real numbers such that $(T(x_n))$ is also u.d. mod 1. (For the definition of u.d. mod 1 sequences, cf. [12].) Precisely, for some convex monotone transformation $T:\mathbb{R}\to\mathbb{R}$, there exists a sequence (x_n) such that $\langle\mu_n(x_n)\rangle\to\lambda_{\mathbb{T}}$, $\langle\mu_n(T(x_n))\rangle\to\lambda_{\mathbb{T}}$ weakly. In fact, it follows directly from Weyl's criterion [12, Chapter 1, Theorem 2.1] that for every linear transformation with a non-zero slope, (x_n) is u.d. mod 1 if and only if $(T(x_n))$ is u.d. mod 1. Also, for $T(x)=e^x$, both (αn) and $(T(\alpha n))$ are u.d. mod 1, for almost all irrational numbers α [12, Chapter 1, Corollary 4.1]. However, it remains open whether (a^n) is u.d. mod 1, for some specific positive number a, for instance, when $a=e,\ \pi$ or even as simple as $3/2$ [12, p. 36].

Then a natural analogous question arises: for a given convex monotone transformation $T:\mathbb{R}\to\mathbb{R}$, does there always exist a probability measure μ on $\mathbb{R}$ such that $\langle\mu\rangle=\left\langle\mu\circ T^{-1}\right\rangle=\lambda_{\mathbb{T}}$? In other words, does there exist a CIUPM for T? As we will show in this paper, though in the discrete version it is trivial that for any u.d. mod 1 sequence (x_n), $(T(x_n))$ is u.d. mod 1 for any linear T (with a non-zero slope), it may not be as trivial to show the existence of a CIUPM for a linear transformation T as shown later in Section 3.

This work, as a first try, answers the question for (the trivially convex) linear transformations T. For nonlinear convex transformations, like the exponential transformation, the problem is more difficult in that such CIUPMs, if they exist, cannot be easily solved by their densities as for the linear case in this paper. Indeed, even for a piecewise linear transformation (for instance $T(x)=x\mathbb{1}_{]-\infty,0]}+\sqrt{2}x\mathbb{1}_{]0,+\infty[}$), situation becomes much more complicated than the linear case. This will be illustrated more clearly when solving the equations for the densities of a CIUPM in the proof of the main result. Except for the existence of a CIUPM, we present a threshold result characterizing how "slim" a CIUPM can be: For a linear transformation T with non-zero slope, there exists a CIUPM μ for T if and only if $\operatorname{diam}(\operatorname{supp}\mu)\geq c$ for some positive constant c depending only on the slope of the linear transformation, where $\operatorname{supp}\mu$ is the support of μ (i.e., the smallest closed subset in $\mathbb{R}$ of full μ measure). Moreover, the CIUPM is unique up to translation if $\operatorname{diam}(\operatorname{supp}\mu)=c$.

Let us mention some related works on invariant measures for "almost" linear transformations on $[0,1]$. Kopf [11] gave a formula for the densities of invariant measures for piecewise linear transformations on $[0,1]$. Góra [2,3] found an explicit formula for the densities of invariant measures for arbitrary eventually expanding piecewise linear transformations whose slopes are not necessarily the same on $[0,1]$.

For $\alpha\in\mathbb{R}$, $\beta\neq 0$, define $\langle T_{\alpha,\beta}\rangle:\begin{cases}\mathbb{T}\to\mathbb{T},\\ x\mapsto\langle\beta x+\alpha\rangle.\end{cases}$ For $0\leq\alpha<1$ and $\beta>1$, Parry [14] gave an explicit formula for the unique invariant measure. Halfin [4] showed this invariant measure is positive. Hofbauer [5–8] proved that this measure is absolutely continuous (w.r.t. λ), its entropy equals $\log\beta$, and its support is a finite union of intervals; he also showed the uniqueness of invariant measures with maximal entropy and determined the region of (β,α)-plane where $\operatorname{supp}\mu\subset[0,1]$. Faller and Pfister [1] studied normal points for $\langle T_{\alpha,\beta}\rangle$.