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Explicit terms in the small volume expansion of the shift of Neumann Laplacian eigenvalues due to a grounded inclusion in two dimensions

Alexander Dabrowski

Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland

A R T I C L E I N F O

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A B S T R A C T

The first terms in the small volume asymptotic expansion of the shift of Neumann Laplacian eigenvalues caused by a grounded inclusion of area ε^2 are derived. A novel explicit formula to compute them from the capacity, the eigenvalues and the eigenfunctions of the unperturbed domain, the size and the position of the inclusion, is given. The key step in the derivation is the filtering of the spectral decomposition of the Neumann function with the residue theorem. As a consequence of the formula, when a bifurcation of a double eigenvalue occurs (as for example in the case of a generic inclusion inside a disk) one eigenvalue decays like $O(1/\log\varepsilon)$, the other like $O(\varepsilon^2)$.

1. Introduction

Consider a planar domain Ω and let ω^2 be an eigenvalue of the negative Laplacian on Ω with homogeneous Neumann boundary conditions. Suppose a small inclusion $D = z + \varepsilon B$ (where $z \in \Omega$, $|B| = |\Omega|$, and ε is small) is inserted inside Ω. This may cause the eigenvalue ω_ε^2 of the perturbed domain $\Omega \setminus D$ (with Neumann condition on $\partial\Omega$ and Dirichlet on ∂D) to vary in value or in multiplicity with respect to ω^2. Asymptotic formulae of the perturbation with respect to the size of the inclusion have been derived in the '80s in [11] and [5] (see also [10, Chapter 9]). In particular it has been shown that if ω^2 is simple and u is the associated L^2-normalized eigenfunction, the perturbation is singular and

$$\omega_\varepsilon^2 - \omega^2 = -\frac{2\pi |u(z)|^2}{\log\varepsilon} + o\left(1/\log(\varepsilon)\right). \tag{1}$$

More recently, Gohberg–Sigal theory for meromorphic operators applied to the integral equation formulation of the eigenvalue problem has led to new results (see [4] and [2]). In this paper we elaborate on these results

E-mail address: aldabrow@ethz.ch.

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to further improve (1), by calculating explicitly the terms up to $O(\varepsilon^2)$ and by generalizing it to the case of multiplicity 2. As a consequence of our derivation, for perturbed eigenvalues $\omega_{\varepsilon,1}^2 < \omega_{\varepsilon,2}^2$ split from a double eigenvalue ω^2 of the original domain Ω, it holds

$$\begin{aligned} \omega_{\varepsilon,2} - \omega &= -\frac{C_1}{\log(\varepsilon) + C_2} + O(\varepsilon^2), \\ \omega_{\varepsilon,1} - \omega &= O(\varepsilon^2), \end{aligned}$$

where C_1 and C_2 do not depend on ε and can be explicitly calculated from the capacity, the eigenvalues, and the value at z of the eigenfunctions of Ω.

More in detail the structure of the paper is as follows. After introducing in Section 1.1 the precise setting of the problem and the notation, in Section 1.2 we recall the equivalent formulation of the Laplacian eigenvalues as characteristic values of an appropriate integral operator. An asymptotic expansion of this integral operator can be obtained by expanding in Taylor series the free space fundamental solution. Gohberg–Sigal theory then provides a link between eigenvalues' shifts and the traces of these integral operators through power sum polynomials. In the core Section 2, explicit terms for the small volume expansion of these power sum polynomials are derived by using properties of layer potentials. The key step in this derivation is the filtering of the spectral decomposition of the Neumann function using the residue theorem to obtain geometric-like series which can be summed. A tentative proposal for formal automated computation of higher order coefficients is given in Section 2.3. Finally in Section 3 some interesting consequences for special cases and a brief validation with numerical experiments are provided.

1.1. Main tools and notation

The eigenvalue problem Let Ω be an open, bounded and connected subset of $\mathbb{R}^2$ with C^1 boundary. It is well known that the eigenvalues of the negative Laplacian on Ω with Neumann boundary condition are non-negative, have finite multiplicity and can be arranged in an increasing divergent sequence

$$0 = \omega_0^2 < \omega_1^2 < \omega_2^2 < \cdots < \omega_k^2 \to \infty.$$

For each index i, let m_i be the multiplicity of ω_i^2. We choose the associated eigenfunctions $u_{i,1}, \dots, u_{i,m_i}$ to be orthonormal in L^2. For any i, j we thus have

$$\begin{cases} (\Delta + \omega_i^2) u_{i,j} = 0 & \text{in } \Omega, \\ \dfrac{\partial u_{i,j}}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\int_\Omega u_{i,j} u_{k,l} = \begin{cases} 1 & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

We will often use the vector notation

$$U_i := (u_{i,1}, \dots, u_{i,m_i}). \tag{2}$$

Free space fundamental solution The free space fundamental solution for Helmholtz equation $(\Delta+\omega^2)u = 0$ is a function Γ_ω s.t. for any $x, y \in \mathbb{R}^2$, it holds

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