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Asymptotic expansions of the Helmholtz equation solutions using approximations of the Dirichlet to Neumann operator

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A R T I C L E I N F O

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A B S T R A C T

This paper is concerned with the asymptotic expansions of the amplitude of the solution of the Helmholtz equation. The original expansions were obtained using a pseudo-differential decomposition of the Dirichlet to Neumann operator. This work uses first and second order approximations of this operator to derive new asymptotic expressions of the normal derivative of the total field. The resulting expansions can be used to appropriately choose the ansatz in the design of high-frequency numerical solvers, such as those based on integral equations, in order to produce more accurate approximation of the solutions around the shadow and the deep shadow regions than the ones based on the usual ansatz.

1. Introduction

Studying the Helmholtz equation at the high-frequency regime is fundamental in both the theoretical understanding of the corresponding solutions and the derivation of appropriate numerical schemes. The well-know asymptotic expansions developed by Melrose and Taylor [40] have significantly contributed in this matter and were the key in the design of several high-frequency integral equation methods. Indeed, integral equation methods are very efficient and widely used in the solution of acoustic scattering problems (see e.g. [23,21] and the references therein). However, the resulting linear systems are dense, ill-conditioned and with large size in particular when the frequency increases. Several effective strategies have been proposed to overcome these difficulties [23,21,3,15,20,7–9,36,42,17,5,31,43,45,16,13,14,12]. Despite this significant progress, integral formulations are limited at higher frequencies since the numerical resolution of field oscillations can easily lead to impractical computational times. This is why hybrid numerical methods based on a combination of integral equations and asymptotic methods have found an increasing interest for the solution of high-frequency scattering problems. Indeed, the methodologies developed in this connection that specifically

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concern scattering from a smooth convex obstacle were first introduced in [1,2]. Several other works followed these [19,28,4,18,25,35,30,27,26] and mainly consist of improving and analyzing this kind of numerical algorithms in single and multiple scattering configurations. All these methods are mainly based on construction of an appropriate ansatz for the solution of integral equations in the form of a highly oscillatory function of known phase modulated by an unknown slowly varying envelope, which is expected to generate linear systems quasi-independent of the frequency.

The high-frequency integral equation methods mentioned above use the asymptotic expansions developed in the well-known paper by Melrose and Taylor [40] in the context of convex obstacles. From these expansions, an ansatz is derived and incorporated into integral equations to eliminate the highly oscillatory part of the unknown which usually corresponds to the physical density, normal derivative of the total field, computed on the surface of the obstacles. This surface is decomposed into three regions, the illuminated and shadows regions in addition to the deep shadow one. Each region is then numerically treated differently and the ansatz is set in general on the illuminated one. Although carefully designed, the aforementioned high-frequency integral equation formulations result in ill-conditioned matrices that limit the numerical accuracy of the approximate solutions. One explanation lies in the fact that the rapidly decaying behavior of the unknown density in the deep shadow regions is not incorporated into the approximation spaces as it is not intrinsic to the chosen ansatz. Generally speaking, it is not clear how to extract all the information needed from the leading term in the expansion given in [40], which restricts the construction of the ansatz.

In this paper, we derive new expansions of the normal derivative of the total field using approximations of the Dirichlet to Neumann (DtN) operator. The original expansions employed a pseudo-differential decomposition of the DtN operator, and the related analysis focuses on the behavior of this field around the shadow boundary which leads to a corrected formula for the Kirchhoff approximation around this region [40]. However, it has been shown that these expansions are valid in the entire surface of the obstacles [25, 28]. Here, we choose first and second order approximations of the DtN operator of the Bayliss–Turkel type [11,38]. These conditions were designed to deal with the infinite aspect of the computational domain for scattering problems. They were also employed in the design of the On-Surface Radiation Conditions [6]. Other approximations such as those developed in [29] can also be adapted to our analysis without specific difficulties. To obtain these new expansions, we follow a similar procedure to the one given in [40]. Briefly, it consists of first finding the kernel of a certain operator, which allows the computation of its amplitude, and then use the stationary phase method to get the final expansions around the shadow boundary. In this case, we can use some of the results derived by Melrose and Taylor [40] in our analysis. The resulting expansions can then be used to appropriately build an ansatz that contains the expected behavior of the solution in the three regions, namely, the illuminated and the deep shadow regions in addition to the shadow boundaries. This provides an improvement over the usual ansatz that behaves like Kirchhoff approximations, meaning that the corresponding solutions are accurate mostly in the illuminated regions.

This paper is organized as follows. After reviewing the functional setting needed for this analysis, we state the problem and explain our choice, regarding the approximation of the DtN operator, in the second section. The two following sections are, respectively, devoted to the derivation of asymptotic expansions in the context of first and second order approximations of the DtN operator. The last section is reserved for some conclusions.

In this work, we will use the following functional spaces (for more details, see for instance [44,22]). Let U be an open bounded set of $\mathbb{R}^n$.

- $D(U)$: space of smooth test functions with compact support, from U to $\mathbb{R}^n$.
- $D'(U)$: space of distributions.
- $S(\mathbb{R}^n)$: Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^n$.
- $S'(\mathbb{R}^n)$: space of tempered distributions, which is the dual space of $S(\mathbb{R}^n)$.
- $\mathcal{E}'$: space of compactly supported distributions.

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