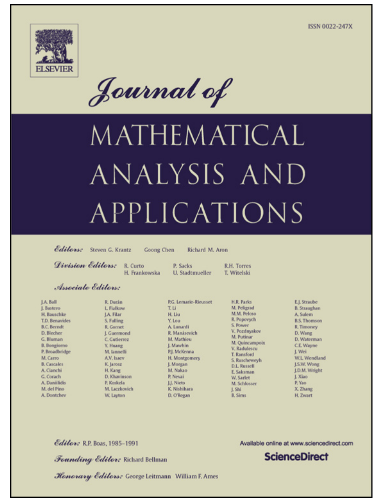

Remarks on Li–Yau inequality on graphs

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Abstract

In this paper, we study Li-Yau gradient estimates for the solutions u to the heat equation $\partial_t u = \Delta u$ on graphs under the curvature condition $CD(n, -K)$, which is introduced by Bauer et al. in [5]. As applications, we derive Harnack inequalities and heat kernel estimates on graphs. Also we present a type of Hamilton gradient estimates.

Keywords: Li-Yau inequality, Heat equation, Curvature, Heat kernel
AMS Classification Subjects 2010: 05C81 53C21 35K08

1 Introduction

In their celebrated work, Li and Yau [11] proved an upper bound on the gradient of positive solutions to the heat equation, called Li-Yau inequality. This inequality is a very powerful tool to study estimation of heat kernels. More precisely, in its simplest form, it asserts that, for an n-dimensional compact Riemannian manifold with non-negative curvature, if u is a positive solution to the heat equation $\partial_t u = \Delta u$, then

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}.$$

Many generalizations of this inequality have been developed, see [7, 4, 10, 12, 15, 2, 3] and references therein.

Recently, Bauer et al [5] prove a discrete version of Li-Yau inequality on graphs via introducing a new notion of curvature, a type of chain rule formula for graph and a discrete version of maximum principle, see also [13, 14] for recent progress in this direction. More precisely,

Theorem (Due to [5]) Let $G = (V, E)$ be a finite graph satisfying $CDE(n, -K)$ with $K \geq 0$, and let u be a positive solution to the heat equation on G. Then for fixed $0 < \alpha < 1$, we have for all $t > 0$,

$$\frac{(1-\alpha)\Gamma(\sqrt{u})}{u} - \frac{\frac{\partial}{\partial t}(\sqrt{u})}{\sqrt{u}} \leq \frac{n}{2(1-\alpha)t} + \frac{Kn}{\alpha}.$$

In particular, if $K = 0$, we can take $\alpha = 0$.

Meanwhile, for the Laplacian on the manifolds with negative curvature, the parameter α can be replaced by some function of the time t, for example,

Theorem (Due to [4, 10]) Let M be a complete Riemannian manifold with dimension n. Assume $Ricci(M) \geq -K$ with $K \geq 0$. For any solution u to the heat equation $\partial_t u = \Delta u$, we have for $t > 0$,

$$|\nabla \log u|^2 - \left(1 + \frac{2}{3}Kt\right)(\log u)_t \leq \frac{n}{2t} + \frac{nK}{2}\left(1 + \frac{1}{3}Kt\right),$$

and

$$|\nabla \log u|^2 - \left(1 + \frac{\sinh(Kt)\cosh(Kt) - Kt}{\sinh^2(Kt)}\right)(\log u)_t \leq \frac{nK}{2}\left(1 + \coth(Kt)\right).$$

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