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Numerical ranges of products of two positive contractions

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ABSTRACT

Let A and B be positive contractions on a Hilbert space. It was known before that the numerical range $W(AB)$ of their product is contained in the rectangular region $[-1/8, 1] \times [-1/4, 1/4]$. In this paper, we determine exactly when $W(AB)$ touches on its four sides. This follows from the more general information on a containing region of $W(AB)$ and on when they share a common tangent line. We also give an example of A and B on three-dimensional space for which $W(AB)$ is not symmetric with respect to the x-axis.

For a bounded linear operator T on a Hilbert space H, its *numerical range* is, by definition, the subset $\{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$ of the complex plane, where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are the inner product and its associated norm in H, respectively.

The purpose of this paper is to give a more specific information on the numerical range of the product of two positive contractions. Indeed, let A and B be two such operators, that is, they satisfy $0 \le \langle Ax, x\rangle, \langle Bx, x\rangle \le 1$ for all unit vectors x in H. It was proven by W.G. Strang [5] before that $W(AB)$ is contained in the rectangular region $[-1/8, 1] \times [-1/4, 1/4]$ (cf. also [4, Proposition 1] and [1, Theorem 1.1]). The extreme values $-1/8$, 1, $-1/4$ and $1/4$ are sharp as seen by the examples of $(A, B) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{bmatrix}\right)$ for $-1/8$, $([1], [1])$ for 1, and $\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\right)$ for $\pm 1/4$. In Theorem 1 below, we give a containing region $\triangle$ of $W(AB)$, which is minimal in the sense that when their boundaries $\partial\triangle$ and $\partial W(AB)$ share a common tangent line, A and B must have certain specific matrices as their direct summands. It follows as a corollary that $W(AB)$ touches on the vertical line $\mathrm{Re}\, z = -1/8$ if and only if A and B are simultaneously unitarily similar to operators of the form $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \oplus A_1$ and $\begin{bmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{bmatrix} \oplus B_1$, respectively (cf. Corollary 2 (b)). The other matrices given above play the analogous roles of the direct summands for the other three sides of the rectangular region (cf. Corollary 2 (c) and (d)).

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At the end of this paper, we consider the symmetry property of $W(AB)$ relative to the x-axis. Although the region from Theorem 1 (a) and some special cases proven in Proposition 4 seem to suggest this to be the case, Example 5 shows that this is not always so.

For a subset $\triangle$ of the real line, we distinguish between $\sup\triangle$ and $\max\triangle$: the latter indicates that the value of the former is attained by some element of $\triangle$. If K is a set of vectors in a Hilbert space, we use $\bigvee K$ to denote the (closed) subspace generated by the vectors in K. For an operator A, $\operatorname{Re} A$ and $\operatorname{Im} A$ denote its *real part* $(A+A^*)/2$ and *imaginary part* $(A-A^*)/(2i)$, respectively. Pairs of operators (A,B) and (C,D) are *unitarily similar* if there is a unitary operator U such that $U^*AU=C$ and $U^*BU=D$. Our reference for general properties of operators is [2] and, in particular, for the numerical range its Chapter 22.

The next theorem is the main result of this paper. It gives a sharp containing region of the numerical range of the product of two positive contractions.

Theorem 1. *Let A and B be positive contractions on a Hilbert space H, and let t in $[-\pi,\pi]$.*

(a) *We have*

$$\sup W\big(\operatorname{Re}(e^{it}AB)\big) \le \begin{cases} \cos t & \text{if } |t|\le \frac{\pi}{3}, \\ \frac{1}{4(1-\cos t)} & \text{if } \frac{\pi}{3}\le |t|\le\pi. \end{cases}$$

(b) *If $|t|\le\pi/3$, then $\max W\big(\operatorname{Re}(e^{it}AB)\big)=\cos t$ if and only if (A,B) is unitarily similar to $([1]\oplus A_1,[1]\oplus B_1)$ for some operators A_1 and B_1.*

(c) *If $\pi/3\le|t|\le\pi$, then $\max W\big(\operatorname{Re}(e^{it}AB)\big)=1/\big(4(1-\cos t)\big)$ if and only if (A,B) is unitarily similar to*

$$\left(\frac{1}{2(1-\cos t)}\begin{bmatrix}1 & \sqrt{1-2\cos t}\\ \sqrt{1-2\cos t} & 1-2\cos t\end{bmatrix}\oplus A_2,\ \frac{1}{2(1-\cos t)}\begin{bmatrix}1 & e^{it}\sqrt{1-2\cos t}\\ e^{-it}\sqrt{1-2\cos t} & 1-2\cos t\end{bmatrix}\oplus B_2\right)$$

for some operators A_2 and B_2.

Note that (a) of the preceding theorem gives a containing region $\triangle_1$ of $W(AB)$ via the supporting lines of $W(AB)$ which are perpendicular to the ray from the origin having angle $-t$ from the positive x-axis. In contrast, [5, Theorem III] yields another containing region $\triangle_2$ of $W(AB)$ with boundary curve in polar coordinate

$$r=\left(\frac{\cos(\pi/3)}{\cos\big((\pi-|\theta|)/3\big)}\right)^3,\quad -\pi\le\theta\le\pi. \tag{1}$$

An immediate consequence of Theorem 1 is the following corollary.

Corollary 2. *Let A and B be positive contractions on a Hilbert space. Then*

(a) *$W(AB)$ is contained in the rectangular region $[-1/8,1]\times[-1/4,1/4]$,*
(b) *the following are equivalent:*
   (i) *$W(AB)\cap\{z\in\mathbb{C}:\operatorname{Re}z=-1/8\}\ne\emptyset$,*
   (ii) *(A,B) is unitarily similar to $\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\oplus A_1,\begin{bmatrix}1/4&\sqrt{3}/4\\ \sqrt{3}/4&3/4\end{bmatrix}\oplus B_1\right)$ for some operators A_1 and B_1,*
   (iii) *AB is unitarily similar to $\begin{bmatrix}1/4&\sqrt{3}/4\\0&0\end{bmatrix}\oplus C$ for some operator C,*