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Invariant measures for multivalued semigroups

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A B S T R A C T

In this work we extend the concept of an invariant measure for a multivalued semigroup and, when it has a global attractor, we give different, but equivalent, definitions for such a measure. As a consequence we can apply the Birkhoff Ergodic Theorem to conclude that time averages converge almost everywhere to the spatial average.

1. Introduction

There have been various attempt to define invariant measures for multivalued applications. The most precocious introduced by Vershik in [35], followed by a definition from Aubin, Frankowska and Lasota in [8] and, because of the increasing attention paid to multivalued dynamical systems during the 80's and 90's, some other definitions have been proposed, as we can see in [31] and [3–6]. In [5] and [32], authors prove the equivalence, under appropriated conditions, of the main definitions in the discrete dynamical system context, with some applications to ordinary differential equations in [3–6].

The study of invariant measures for well posed partial differential equations is done in Wang, Luckaszewicz, Robinson and Real, and Checkroun's works, respectively in [38], [29,30] and [17] for autonomous dissipative problems. Invariant measures for non-autonomous well posed problems have been studied by Luckaszewicz and Robinson in [28].

We can define an *invariant measure for a semigroup* $\{S(t)\}_{t\in\mathbb{R}^+}$ on a complete metric space X, as a Borel probability measure μ satisfying

$$\mu(A) = \mu(S(t)^{-1}(A)) \quad \forall\, t \geq 0 \text{ and for any Borel subset } A \subset X.$$

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Equivalently we can say that μ is invariant for $S(\cdot)$ if

$$\int_X \varphi(x)\mu(dx) = \int_X \varphi(S(t)x)\mu(dx)$$

for any $t \geq 0$ and φ continuous and bounded on X, [17,28].

In [30] and [28] the authors define a family of Borel probability measures $\{\mu_t\}$ as invariant for a non-autonomous system $\{U(t,\tau)\}_{t\geq\tau}$ if

$$\mu_t(B) = \mu_\tau\left(U(t,\tau)^{-1}(B)\right) = \left(U(t,\tau)_*(\nu_\tau)\right)(B), \quad t \geq \tau \text{ (in [30])},$$

or equivalently,

$$\int_{A(t)} \varphi(x)\mu_t(dx) = \int_{A(\tau)} \varphi(U(t,\tau)x)\mu_\tau(dx) \text{ for each } t \geq \tau \text{ and each } \varphi \in \mathcal{C}(X), \text{ (in [28])}.$$

For multivalued infinite dimensional dynamical systems, the main references related with invariant measures are connected to Navier–Stokes problems and the concept of statistic solutions, introduced by Foias in [20], following Hopf and Prodi previous ideas respectively found in [23] and [33]. Later Vishik and Fursikov introduced in [37] a new concept of statistical solutions, defined on a trajectory space determined by an evolution problem. More recently, Foias, Rosa and Temam, focusing on Navier–Stokes problems, improved this concept, defining a measure on a trajectory space, the Vishik–Fursikov measure, whose projection on the phase space at each time t generates a projected statistic solution, called Vishik–Fursikov statistic solution, which recover the former concept proposed by Prodi and Foias (see [22] and references therein). The particular case when the Vishik–Fursikov statistic solutions are stationary coincides, as we are going to prove, with an invariant measure for the evolution system associated with the above mentioned trajectory space. It is worth to mention the works [12,14] and [13] where we can find the first ideas of an abstract theory on statistic solutions for more general evolution problems. The equivalence between invariant measures and statistic solutions are pointed in [16] and [25] for a well posed version of the Navier Stokes problem.

In this work, we extend to multivalued evolution problems the main definitions of invariant measures and, under suitable conditions, we prove the equivalence of such definitions. As a relevant consequence, we can apply the Birkhoff Ergodic Theorem to multivalued dissipative evolution problems in order to conclude the convergence of time averages almost everywhere with respect to an invariant measure.

2. Basic definitions, notations and terminologies

For a nonempty metric space X, we use the notations $\mathbb{B}(X)$ and $\mathbb{P}(X)$ to indicate the Borel σ-algebra on X and the set of Borel probability measures on X respectively. $\mathbb{P}(X)$ is embedded with the weak topology, the coarsest which makes continuous the function $\mu \mapsto \int_X f(x)\mu(dx)$ for each $f \in C(X,\mathbb{R})$. The convergence of a sequence $\{\mu_n\}$ in $\mathbb{P}(X)$ is given by

$$\mu_n \to \mu \Leftrightarrow \int_X f(x)\mu_n(dx) \to \int_X f(x)\mu(dx)$$

for each $f \in C(X,\mathbb{R})$. Since X is a compact metric space, $\mathbb{P}(X)$ is a compact metrizable space. On $\mathbb{P}(X)$ we consider the Hutchinson metric given by

$$d(\mu_1,\mu_2) = \sup\left\{\int_X f(x)\mu_1(dx) - \int_X f(x)\mu_2(dx)\right\}, \tag{2.1}$$