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On the energy decay for dissipative nonlinear wave equations in one space dimension

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A R T I C L E I N F O

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A B S T R A C T

The energy decay problem is studied for the nonlinear dissipative wave equation in one space dimension. It is shown by Mochizuki and Motai [6] that the decay rate is at least logarithmic when the exponent of the nonlinearity is greater than one and less than three. In this paper, an improvement is found which implies a polynomial decay rate for the same range of exponents.

1. Introduction

The long time behavior of energy is one of the main problems in dissipative evolution equations. In the present paper, we consider the wave equation with nonlinear damping

$$u_{tt} - u_{xx} + |u_t|^{p-1}u_t = 0, \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

and initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where $p > 1$ and $(u_0, u_1) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ satisfy $\operatorname{supp}(u_0, u_1) \subset [-R, R]$. It is well known that the global well-posedness is established by Lions and Strauss [3] for any initial data $u_0 \in H^{s+1}(\mathbb{R})$, $u_1 \in H^s(\mathbb{R})$ and $0 \le s \le 1$. The asymptotic behavior of solutions is, however, far from well understood.

Let us briefly discuss the known results concerning energy decay and non-decay for equation (1.1) in general space dimensions, i.e.,

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$$u_{tt} - \Delta u + |u_t|^{p-1} u_t = 0, \qquad x \in \mathbb{R}^n, \ t > 0, \tag{1.3}$$

where $n \geq 1$. We define the energy of solutions to (1.3) by

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} (u_t^2 + |\nabla_x u|^2) dx \tag{1.4}$$

and easily check that the following energy identity holds:

$$E(t) = E(s) - \int_s^t \|u_\tau(\tau)\|_{p+1}^{p+1} d\tau \quad \text{for } 0 \leq s < t. \tag{1.5}$$

Thus, $E(t)$ is a decreasing function of $t > 0$. Our main interest in this paper is to determine the decay rate, if any, of (1.4) as $t \to \infty$ and u is a solution to (1.1).

The energy decay problem for (1.3) is studied in a pioneering paper by Mochizuki and Motai [6]. On the one hand, the estimate

$$E(t) \leq C\{\log(2 + t)\}^{-\gamma}$$

is established applying a new multiplier method for $1 < p < 1 + 2/n$ and $\gamma > 0$ depending on p. On the other hand, the non-decay on energy is shown using dispersive estimates for the wave equation when $p > 1 + 2/(n - 1)$ and $n \geq 2$. Matsuyama [5] has extended the non-decay results to convex exterior domains and proved that sufficiently small and regular solutions are asymptotically free if $n = 3$. The first polynomial decay estimate of energy is obtained by Todorova and Yordanov [8]:

$$E(t) \leq Ct^{-a},$$

as $t \to \infty$, for $1 < p < 1 + 1/(n + 1)$ and $n \geq 3$ with some $a > 0$ depending on p and n. Although the latter work gives no decay estimates of total energy in dimensions $n \leq 2$, it shows that the “exterior energy” restricted to $|x| > t^{1/2}$ decays at a fast polynomial rate as $p \to 1$. In this paper, such “parabolic effects” contribute to an improvement in the polynomial decay rate of $E(t)$ for equation (1.1).

For small and smooth initial data, Katayama, Matsumura and Sunagawa [2] have obtained the logarithmic decay of energy in the case of $p = 3$ and $n = 2$. Their method is based on the commuting vector fields for $\partial_t^2 - \Delta$.

It is important to mention that the long time behavior of nonlinear dissipative wave equations like (1.3) is quite difficult to study; the presence of nonlinearity excludes methods in the frequency space (Fourier transform), while the lack of mass term and non-divergence form of the dissipation limits the effectiveness of methods in the physical space (multipliers and divergence theorem). In contrast, the wave equation with linear damping admits precise L^q estimates obtained by Matsumura [4] through the Fourier transform representation of solutions. Another example is the nonlinear dissipative Klein–Gordon equation where the mass term is used by Nakao [7] to control the L^2 norm and derive a polynomial decay rate for the energy of solutions.

Let us return to the main problem (1.1), (1.2). Except for the logarithmic decay estimates in [6], there seem to be no results concerning the long time behavior of its solutions. Here we address this issue by establishing polynomial decay estimates for $E(t)$ in (1.4) when $1 < p < 3$. The strategy of our proof is to combine the weighted energy of Mochizuki and Motai [6] with the $W^{1,1}(\mathbb{R}) \times L^1(\mathbb{R})$ estimate of Haraux [1] which improves a crucial L^{p+1} estimate and provides a uniform L^∞ estimate for solutions to (1.1), (1.2); see Proposition 2.1 and 2.3 below. We also rely on the “exterior energy” estimate in Todorova and Yordanov [8], Proposition 2.5, to maximize the decay rate when p is close to 1. Our main result is the following: