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Controllability of a 2D quantum particle in a time-varying disc with radial data

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A B S T R A C T

In this article we consider a 2-D quantum particle confined in a disc whose radius can be deformed continuously in time. We study the problem of controllability of such a quantum particle via deformations of the initial disc, i.e., when we set the time-dependent radius of the disc to be the control variable. We prove that the resulting system is locally controllable around some radial trajectories which are linear combinations of the first three radial eigenfunctions of the Laplacian in the unit disc with Dirichlet boundary conditions. We prove this result, thanks to the linearisation principle, by studying the linearised system, which leads to a moment problem that can be solved using some results from Nonharmonic Fourier series. In particular, we have to deal with fine properties of Bessel functions.

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1. Introduction

1.1. Physical background

We consider a d-dimensional quantum particle, for $d \geq 1$, of mass m, under no external forces. According to Quantum Mechanics, the state of such a particle can be described by a complex-valued wave-function (see [6, Secs. 2.2.1, 2.2.3])

$$\psi : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{C}, \qquad \text{with} \int\limits_{\mathbb{R}^d} |\psi(t,x)|^2 \, \mathrm{d}x = 1,^{1} \quad \forall t \in \mathbb{R}^+,$$

satisfying the Schrödinger equation

$$i\partial_t \psi = -\frac{\hbar}{2m} \Delta_x \psi, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

where $\hbar$ stands for the normalised Planck constant. In some instances (e.g., potential wells [6, Sect. 4.3.4]), it is possible to confine the dynamics of a quantum particle within a region of the space, namely a regular open set $\Omega \subset \mathbb{R}^d$, which leads to a boundary-value problem for the associated wave-function, of the form

$$\begin{cases} i\partial_t \psi = -\frac{\hbar}{2m} \Delta_x \psi, & (t,x) \in \mathbb{R}^+ \times \Omega, \\ \psi = 0, & (t,x) \in \mathbb{R}^+ \times \partial\Omega, \end{cases}$$

and the condition

$$\int\limits_{\Omega} |\psi(t,x)|^2 \, \mathrm{d}x = 1, \quad \forall t \in \mathbb{R}^+.$$

This allows to consider time-dependent confinement regions, namely a family of smooth open sets $\{\Omega(t)\}_{t \geq 0}$, varying continuously with respect to time, within which the particle is confined. This question has attracted attention in Physics literature, as the works [15,29,23] or the survey [18] account for.

In terms of the wave function, a quantum particle confined in $\{\Omega(t)\}_{t \geq 0}$ must satisfy

$$\int\limits_{\Omega(t)} |\psi(t,x(t))|^2 \, \mathrm{d}x = 1, \quad \forall t \in \mathbb{R}^+, \tag{1.1}$$

and the Schrödinger equation

[1] The measure $|\psi(t,x)|^2 \, \mathrm{d}x$ is interpreted as a probability density, which explains the constraint.