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Monotonicity of facet numbers of random convex hulls

Gilles Bonnet [a,*], Julian Grote [a], Daniel Temesvari [a], Christoph Thäle [a], Nicola Turchi [a], Florian Wespi [b]

[a] *Faculty of Mathematics, Ruhr University Bochum, Germany*
[b] *Institute of Mathematical Statistics and Actuarial Science, University of Bern, Switzerland*

A R T I C L E I N F O

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A B S T R A C T

Let $X_1, \dots, X_n$ be independent random points that are distributed according to a probability measure on $\mathbb{R}^d$ and let P_n be the random convex hull generated by $X_1, \dots, X_n$ ($n \geq d+1$). For natural classes of probability distributions and by means of Blaschke–Petkantschin formulae from integral geometry it is shown that the mean facet number of P_n is strictly monotonically increasing in n.

1. Introduction and main result

Fix a space dimension $d \geq 2$. For an integer $n \geq d+1$, let $X_1, \dots, X_n$ be independent random points that are chosen according to an absolutely continuous probability distribution on $\mathbb{R}^d$. By P_{n-1} and P_n we denote the random convex hulls generated by $X_1, \dots, X_{n-1}$ and $X_1, \dots, X_n$, respectively. In our present text we are interested in the mean number of facets $\mathbb{E}f_{d-1}(P_{n-1})$ and $\mathbb{E}f_{d-1}(P_n)$ of P_{n-1} and P_n. More specifically, we ask the following monotonicity question:

Is it true that $\mathbb{E}f_{d-1}(P_{n-1}) \leq \mathbb{E}f_{d-1}(P_n)$*?*

This question has been put forward and answered positively by Devillers, Glisse, Goaoc, Moroz and Reitzner [7] for random points that are uniformly distributed in a convex body $K \subset \mathbb{R}^d$ if $d = 2$ and, if $d \geq 3$, under the additional assumptions that the boundary of K is twice differentiable with strictly positive Gaussian curvature and that n is sufficiently large, that is, $n \geq n(K)$, where $n(K)$ is a constant depending on K. Moreover, an affirmative answer was obtained by Beermann [4] if the random points are chosen with respect to the standard Gaussian distribution on $\mathbb{R}^d$ or according to the uniform distribution in the d-dimensional

* Corresponding author.
E-mail addresses: gilles.bonnet@rub.de (G. Bonnet), julian.grote@rub.de (J. Grote), daniel.temesvari@rub.de (D. Temesvari), christoph.thaele@rub.de (C. Thäle), nicola.turchi@rub.de (N. Turchi), florian.wespi@stat.unibe.ch (F. Wespi).

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unit ball for all $d \geq 2$. Beermann's proof essentially relies on a Blaschke–Petkantschin formula, a well known change-of-variables formula in integral geometry. Our aim in this text is to generalize her approach to other and more general probability distributions on $\mathbb{R}^d$. In fact, we will be able to characterize all absolutely continuous rotationally symmetric distributions on $\mathbb{R}^d$ whose densities satisfy a natural scaling property (see (9) below), to which the methodology based on the Blaschke–Petkantschin formula can be applied and for which we can answer positively the monotonicity question posed above for any of these distributions. Moreover, we will apply our results to study similar monotonicity questions for a class of spherical convex hulls generated by random points on a half-sphere, which comprises as a special case the model recently studied by Bárány, Hug, Reitzner and Schneider [3].

To present our main result formally, we introduce four classes of probability measures:

– $\mathbf{G}$ is the class of centered Gaussian distributions on $\mathbb{R}^d$ with density proportional to

$$x \mapsto \exp\Big(-\frac{\|x\|^2}{2\sigma^2}\Big), \qquad \sigma > 0,$$

– $\mathbf{H}$ is the class of heavy-tailed distributions on $\mathbb{R}^d$ with density proportional to

$$x \mapsto \Big(1 + \frac{\|x\|^2}{\sigma^2}\Big)^{-\beta}, \qquad \beta > d/2, \sigma > 0,$$

– $\mathbf{B}$ is the class of beta-type distributions on the d-dimensional centered ball $\mathbb{B}^d_\sigma$ of radius σ with density proportional to

$$x \mapsto \Big(1 - \frac{\|x\|^2}{\sigma^2}\Big)^{\beta}, \qquad \beta > -1, \sigma > 0,$$

– $\mathbf{U}$ comprises the uniform distributions on the $(d-1)$-dimensional centered spheres $\mathbb{S}^{d-1}_\sigma$ with radius $\sigma > 0$.

It will turn out that the classes $\mathbf{G}$, $\mathbf{H}$, $\mathbf{B}$ and $\mathbf{U}$ contain precisely the absolutely continuous rotationally symmetric probability distributions on $\mathbb{R}^d$, whose densities satisfy the natural scaling property (9) below, for which monotonicity of the mean facet number of the associated random convex hulls can be shown by means of arguments based on a Blaschke–Petkantschin formula, see the discussion at the end of Section 4 for further details. In fact, our result shows that even the stronger strict monotonicity holds.

Theorem 1. *Let $X_1, \ldots, X_n \in \mathbb{R}^d$, $n \geq d+1$, be independent and identically distributed according to a probability measure belonging to one of the classes* $\mathbf{G}$, $\mathbf{H}$, $\mathbf{B}$ *or* $\mathbf{U}$. *Then,*

$$\mathbb{E} f_{d-1}(P_n) > \mathbb{E} f_{d-1}(P_{n-1}).$$

It should be emphasized that strict monotonicity of $n \mapsto f_{d-1}(P_n)$ cannot hold pathwise (except for the trivial case $n = d+1$), since the addition of a further random point can reduce the facet number arbitrarily as the additional point might 'see' much more than d vertices of the already constructed random convex hull. For this reason, the expectation in Theorem 1 is essential.

We would also like to remark that monotonicity questions related to the volume of random convex hulls have recently attracted some interest in convex geometry because of their connection to the famous slicing problem. Namely, if K and L are two compact convex sets in $\mathbb{R}^d$ with interior points, let V_K and V_L be the volume of the convex hull of $d+1$ independent random points uniformly distributed in K and L, respectively. One is interested in the question whether the set inclusion $K \subseteq L$ implies the inequality $\mathbb{E}V_K \leq \mathbb{E}V_L$. In