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Groups whose Fourier algebra and Rajchman algebra coincide

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A R T I C L E I N F O

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A B S T R A C T

We study locally compact groups for which the Fourier algebra coincides with the Rajchman algebra. In particular, we show that there exist uncountably many non-compact groups with this property. Generalizing a result of Hewitt and Zuckerman, we show that no non-compact nilpotent group has this property, whereas non-compact solvable groups with this property are known to exist. We provide several structural results on groups whose Fourier and Rajchman algebras coincide as well as new criteria for establishing this property. Finally, we study the relation between groups with completely reducible regular representation and groups whose Fourier and Rajchman algebras coincide. For unimodular groups with completely reducible regular representation, we show that the Fourier algebra may in general be strictly smaller than the Rajchman algebra.

1. Introduction

Over the years there has been considerable interest in studying locally compact groups with completely reducible regular representation, that is, locally compact groups whose regular representation decomposes as a direct sum of irreducible representations (see [3–5,34,45]). By the Peter–Weyl theorem, compact groups are examples of such groups. It may come as a surprise that these are not the only ones. Indeed, in the abelian case it is an easy consequence of the Pontryagin duality theorem that the regular representation of a locally compact abelian group decomposes as a direct sum of irreducible representations if and only if the group is compact.

The study of locally compact groups with completely reducible (also called purely atomic) regular representation is related to the study of certain function algebras associated with the groups. We now describe these algebras (see Section 2 for details).

For a locally compact group G, we let $B(G)$ denote the *Fourier–Stieltjes algebra* consisting of the matrix coefficients of strongly continuous unitary representations of G. The *Fourier algebra* $A(G)$ is the subalgebra of $B(G)$ consisting of the matrix coefficients of the (left) regular representation. It is always the case

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that $A(G) \subseteq B(G) \cap C_0(G)$, and often the inclusion is strict. Here $C_0(G)$ denotes the (complex) continuous functions on G vanishing at infinity. The *Rajchman algebra* $B_0(G)$, which is simply defined as the intersection

$$B_0(G) = B(G) \cap C_0(G),$$

has recently gained renewed interest (see [15,22,24]). But already in 1966, Hewitt and Zuckerman [19] showed that for non-compact abelian groups G, $A(G) \neq B_0(G)$. This generalized a result of Menchoff [35] from 1916, who showed the same for $G = \mathbb{Z}$.

The main objective of the current paper is to study the inclusion $A(G) \subseteq B_0(G)$ and in particular to study when this inclusion can or cannot be an equality:

$$A(G) = B_0(G). \qquad (\star)$$

In [28], it was shown that the minimal parabolic subgroups in real rank one simple Lie groups satisfy ($\star$). This generalized a result of Khalil [21, p. 165] on the $ax + b$ group. Our first contribution is to show that the results from [28] concerning minimal parabolic subgroups in real rank one simple Lie groups do not generalize to higher rank simple Lie groups. The is accomplished in Section 3 by showing that the minimal parabolic subgroup in $\mathrm{SL}(3, \mathbb{R})$ does not satisfy ($\star$).

Non-compact groups which satisfy $A(G) = B_0(G)$ are generally viewed as exceptional, although several examples appeared recently in [28] and [41, Theorem 2.1]. Our second contribution is to show that there are many such groups. We prove the following.

Theorem 4.1. *There exist uncountably many (non-isomorphic) second countable locally compact groups G such that $A(G) = B_0(G)$ and G has no compact subgroups (apart from the trivial group).*

Our next, and probably most substantial, contribution is of a more structural nature. All groups currently known to satisfy ($\star$) match the conditions of [28, Theorem 4] (see Theorem 2.1 below). We show in this paper that there are also groups satisfying ($\star$) that do not match the conditions of that theorem. At the same time, we study how the condition ($\star$) behaves with respect to taking direct products of groups. It is not clear (at least to the author) if the condition ($\star$) is preserved under taking direct products, although we suspect this to be the case. We are, however, able to prove that a finite direct product satisfies ($\star$) provided all the factors are among the groups for which ($\star$) is currently known to hold. We investigate this in Sections 5–7. In particular, in Theorem 7.2 we provide a generalization of [28, Theorem 4].

In [12], Figà-Talamanca studied the Rajchman algebra in relation to having a completely reducible regular representation. He proved that if a unimodular group G satisfies ($\star$), then the regular representation of G is completely reducible. Subsequently, Baggett and Taylor generalized Figà-Talamanca's result to include non-unimodular groups [5, Theorem 2.1]. They proved

Theorem *([5]). If $A(G) = B_0(G)$ for a second countable locally compact group G, then the regular representation of G is completely reducible.*

At some point, people speculated that the converse of the above theorem should hold, that is, that all groups with completely reducible regular representation should satisfy ($\star$). This is not the case, as was shown by Baggett and Taylor [4]. They discovered a non-unimodular group with completely reducible regular representation not satisfying ($\star$). At the same time Baggett and Taylor suggested that the converse of the above theorem should hold for unimodular groups. Our third contribution is to provide an example of a unimodular group whose regular representation is completely reducible, but where ($\star$) fails, thus supplementing the example from [4] and answering (in the negative) the question about unimodularity raised there.

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