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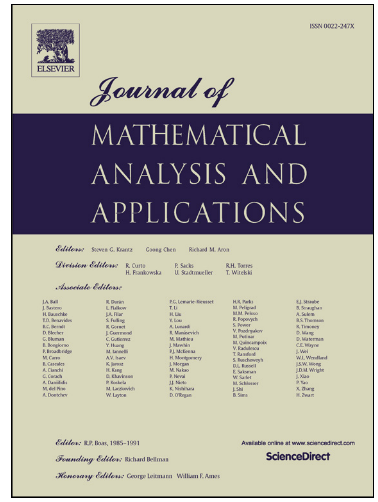


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# NONLINEAR EVOLUTION EQUATIONS THAT ARE NON-LOCAL IN SPACE AND TIME

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Abstract. We deal with a nonlocal nonlinear evolution problem of the form

$$\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)|\overline{v}(y,s)-v(x,t)|^{p-2}(\overline{v}(y,s)-v(x,t))\,dy\,ds=0$$

for $(x,t)\in\mathbb{R}^n\times[0,\infty)$. Here $p\geq 2$, $J:\mathbb{R}^{n+1}\to\mathbb{R}$ is a nonnegative kernel, compactly supported inside the set $\{(x,t)\in\mathbb{R}^{n+1}: t\geq 0\}$ with $\iint_{\mathbb{R}^n\times\mathbb{R}} J(x,t)\,dx\,dt=1$ and $\overline{v}$ stands for an extension of a given initial value $f$, that is,

$$\overline{v}(x,t)=\begin{cases} v(x,t) & t\geq 0,\\ f(x,t) & t<0.\end{cases}$$

For this problem we prove existence and uniqueness of a solution. In addition, we show that the solutions approximate viscosity solutions to the local nonlinear PDE $\|\nabla u\|^{p-2}u_t=\Delta_p u$ when the kernel is rescaled in a suitable way.

## 1. Introduction

Our main goal in this paper is the study of nonlinear evolution problems that are nonlocal both in space and time. Let $F(z)=|z|^{p-2}z$ be a power type nonlinearity and let $J:\mathbb{R}^{n+1}\to\mathbb{R}$, a nonnegative, continuous kernel, compactly supported in the set $\{(x,t)\in\mathbb{R}^{n+1}: t\geq 0\}$ with $\iint_{\mathbb{R}^n\times\mathbb{R}} J(x,t)dxdt=1$. We fix an initial condition $f\in L^\infty(\mathbb{R}^n\times(-\infty,0))$. Our aim is to look for solutions to the nonlocal nonlinear evolution problem

$$(P(J,f))\qquad \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{v}(y,s)-v(x,t))\,dyds=0$$

for $(x,t)\in\mathbb{R}^n\times[0,\infty)$ where we denoted by $\overline{v}$ the extension by $f$ for $t<0$ of a function $v$ defined for $t\geq 0$, that is,

$$\overline{v}(x,t)=\begin{cases} v(x,t) & t\geq 0,\\ f(x,t) & t<0.\end{cases}$$

This paper can be viewed as a natural continuation of [1] where the linear case $p=2$ was considered. Notice that here a solution $u$ verifies a nonlinear mean value formula given by $P(J,f)$.

Our first result deals with existence and uniqueness of solutions. We denote by $\overline{\mathcal{C}}$ the set of uniform continuous functions, and $L^\infty(f)$ stands for the set of bounded functions with norm less or equal than $\|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$.

**Theorem 1.** *Let $J:\mathbb{R}^{n+1}\to\mathbb{R}$ be, nonnegative, continuous and compactly supported in the set $\{(x,t)\in\mathbb{R}^{n+1}:t\geq 0\}$, with $\iint_{\mathbb{R}^n\times\mathbb{R}} J(x,t)dxdt=1$. Let $f\in L^\infty(\mathbb{R}^n\times(-\infty,0))$. Then, there exists a unique $u\in\overline{\mathcal{C}}\cap L^\infty(f)(\mathbb{R}^n\times[0,\infty))$ that solves $P(J,f)$.*

We will use the notation $u$ for a solution with initial datum $f$ and we will say that $u$ solves the problem $P(J,f)$. Note that here we assumed that the kernel is nonnegative and integrable (singular kernels are out of the scope of this paper). This fact together with the choice of $f\in L^\infty(\mathbb{R}^n\times(-\infty,0))$, makes the space $\overline{\mathcal{C}}\cap L^\infty(f)(\mathbb{R}^n\times[0,\infty))$ a natural choice to look for solutions (remark that the integral that appears in $P(J,f)$ is finite under these conditions). Notice that there is a regularizing effect, for $f\in L^\infty(\mathbb{R}^n\times(-\infty,0))$ we obtain a uniformly continuous solution, $u\in\overline{\mathcal{C}}\cap L^\infty(f)(\mathbb{R}^n\times[0,\infty))$. This is due to the fact that we assumed continuity of the kernel $J$.

We have two different proofs of this existence and uniqueness result. The first one is simpler. We just prove first the result for a class of kernels that are compactly supported in the set $\{(x,t)\in\mathbb{R}^{n+1}:\delta\leq t\leq\delta+\gamma\}$, where $\delta$ and $\gamma$ are positive numbers, (this allows us to easily obtain existence and uniqueness of solutions in the strip $t\in[0,\delta)$ and then in $t\in[\delta,2\delta)$, etc.). After that we obtain the result for a general kernel by approximating it with kernels in the previously mentioned class. The second proof is more involved technically and is based on a fixed point argument (we include this proof here since we believe that it has independent interest). This fixed point strategy was used for the

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