J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Partial regularity of a certain class of non-Newtonian fluids ☆

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ARTICLE INFO

Article history:
Received 19 November 2016
Available online xxxx
Submitted by P.G. Lemarie-Rieusset

Keywords:
Non-Newtonian fluids
Partial regularity
Weak solution
Regularity point

ABSTRACT

We study the partial regularity of weak solution to a certain class of non-Newtonian fluids (1.1)–(1.3), the existence of weak solution to such system had been provided by D. Pierre and H. Norbert in [40]. In particular, we prove that the regularity point in Q_T is an open set with full measure, and we obtain a general criterion for a weak solution to be regular in the neighborhood of a given point. In this paper we confined ourselves to considered the case $p = 2$ and space dimension $d = 2$.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with Lipschitz boundary and let $0 < T < \infty$, $Q_T := \Omega \times (-T, 0)$. This paper deals with partial regularity of non-Newtonian fluids

$$\begin{cases} \partial_t u - \operatorname{div} S(Du) + u \cdot \nabla u + D\phi = 0 & \text{in } Q_T, \\ \operatorname{div} u = 0 & \text{in } Q_T. \end{cases} \tag{1.1}$$

The initial condition is given by

$$u|_{t=0} = u_0 \qquad \text{in } \Omega \tag{1.2}$$

and the Dirichlet boundary condition is given by

$$u|_{\partial\Omega \times (-T,0)} = 0. \tag{1.3}$$

☆ This work was supported by the National Natural Science Foundation of China (No. 11271305, 11531010).
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http://dx.doi.org/10.1016/j.jmaa.2017.06.049
0022-247X/

Here $u : Q_T \longrightarrow \mathbb{R}^d$ is the velocity, $\phi : Q_T \longrightarrow \mathbb{R}$ is the pressure, $S(Du) : \mathbb{R}^{d\times d} \longrightarrow \mathbb{R}^{d\times d}$ is the extra stress tensor. In present paper, we considered the stress tensor, which provided by D. Pierre and H. Norbert in [40].

For the general non-Newtonian fluids, there is an amounts of work on the power-law model

$$S(\varepsilon(u)) = \nu_0(\mu + |\varepsilon(u)|^2)^{\frac{p-2}{2}}\varepsilon(u) + \mu_1\varepsilon(u), \tag{1.4}$$

with $\nu_0 > 1, \mu_1 \geq 0$, $p > 1$, $\varepsilon(u)$ denotes the symmetric part of the velocity gradient Du, namely $\varepsilon(u) = \frac{1}{2}(Du + (Du)^T)$. Ladyzhenskaya was one of the first mathematician who considered the modified Navier–Stokes equations and present the mathematical investigation of such model (1.1)–(1.3), (1.4) in 1966, the existence of weak solution to (1.1)–(1.3), (1.4) for $p \geq \frac{3d+2}{d+2}$ had first appeared in [28–30] which is unique for $p > \frac{d+2}{2}$ (see also [31]). The existence of measure-valued solutions for $p > \frac{2d}{d+2}$ was shown in [32,37], for $p > \frac{2d}{d+2}$, the existence of weak solution have been studied in [5,14,33–35]. In [54], J. Wolf constructed a weak solution $u \in L^p(0,T;V_p(\Omega)) \cap C_w(0,T;L^2(\Omega))$ to (1.1)–(1.3), (1.4) for the power with $p > 2\frac{d+1}{d+2}$. In 2014, Bae and Jin [4] study the local in time existence of a weak solution for $\frac{3d}{d+2} < p < 2$ when $d = 2,3$ and the global in time existence of a weak solution for $p \geq \frac{11}{5}$, when $d = 3$. In 2016, Tan and Zhou [50] provided a result about fractional differentiability of the gradient of weak solution Du for $p = 2, d = 2$, that is, $Du \in W^{\alpha,\theta;2}_{loc}(Q_T)$, for all $\theta \in (0, \frac{1}{3}), \alpha \in (0, \frac{1}{2})$. In [51], the author study the higher integrability of weak solution to (1.1)–(1.3), that is

$$u \in L^{p+\varepsilon}(t_1, t_2; W^{1,p+\varepsilon}_{loc}(\Omega)),$$

with $[t_1, t_2] \subset (0,T)$ and $\varepsilon \in \Big(0, \min\{\frac{p(p-1)(d+2)}{2d} - p, \frac{2}{d}\}\Big]$ for $p > \frac{3d+2}{d+2}$. For some other excellent work, one can also refer [20,22,24–26,38,39,43,47,56] and the references therein.

For the variable exponent non-Newtonian fluids of (1.1) (electrorheological fluids), according to the model proposed by Rajagopal and Růžička [41], the system can be looked as the delicate interaction between the electromagnetic fields and the moving fluids:

$$\begin{cases} \text{div } E = 0 \quad \text{curl E} = 0 & \text{in } Q_T, \\ u_t - \text{div } S(z, \varepsilon(u), E) + \text{div } (u \otimes u) + D\phi = f & \text{in } Q_T, \\ \text{div } u = 0 & \text{in } Q_T, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $E(x)$ is the electromagnetic field, $f : Q_T \longrightarrow \mathbb{R}^d$ is the extra force, u, ϕ are defined as (1.1). $S(z, \varepsilon(u), E) : \mathbb{R}^{d\times d} \longrightarrow \mathbb{R}^{d\times d}$ is the extra stress tensor depends in a nonlinear way by $\varepsilon(u)$:

$$S(x, E, \varepsilon(u)) \approx \mu(1 + |\varepsilon(u)|^2)^{\frac{p(|E|^2)-2}{2}}\varepsilon(u) + \text{terms with similar growth},$$

with $p(\cdot) : Q_T \longrightarrow [1,\infty)$ is a given Hölder (log-Hölder) continuous function. These are special fluids characterized by their ability to change in a dramatic way their mechanical properties when in presence of an external electromagnetic field. E. Acerbi et al. in [2] provided the regularity results for parabolic systems related to (1.5): higher integrability, higher differentiability, partial regularity of the spatial gradient, estimates for the (parabolic) Hausdorff dimension of the singular set. For more details, one can also refer [1,3,15–17,42,45,46] and the reference therein.

For the stationary case of (1.1), based on the power-law model (1.4), then there exists a weak solution, this result has been proved independently by Frehse, Málek and Steinhauer [21] ($p \geq \frac{2d}{d+1}$) and M. Růžička [44] ($p > \frac{2d}{d+1}$). In 2007, J. Naumann and J. Wolf [36] (see also [55]) studied the interior differentiability of weak solutions, for $p \in [\frac{2d}{d+1}, 2), d = 2,3$. They first proved $D(u) \in W^{t,\hat{s}}_{loc}(\Omega)$ with $t \in [0, \frac{s(d+2)-3d}{s}]$,