J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Generic points of invariant measures for an amenable residually finite group actions with the weak specification property

Martha Łącka

Faculty of Mathematics and Computer Science, Jagiellonian University in Krakow, ul. Łojasiewicza 6, 30-348 Kraków, Poland

ARTICLE INFO

Article history:
Received 24 May 2017
Available online xxxx
Submitted by U. Stadtmueller

Keywords:
Generic points
Amenable residually finite groups
Specification

ABSTRACT

We prove that every measure invariant for an amenable residually finite group action satisfying the weak specification property has a generic point. This extends the Sigmund theorem and completes the result of Ren.

1. Introduction

Generic points are a powerful tool of ergodic theory, allowing for example to quantify the difference between two measures. Recall that for a T-invariant measure μ, a point x is generic if its orbit is uniformly distributed, that is, for every continuous real valued function f defined on the phase space X one has

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(T^n(x)) = \int_X f\,\mathrm{d}\mu.$$

If a measure is ergodic, then its generic points reflect the most typical behaviour of points from the phase space: it follows from the Birkhoff ergodic theorem that they form a set of the full measure. On the other hand, non-ergodic measures do not have to have any generic points. In fact, there is even a topologically mixing dynamical system with exactly two ergodic measures such that no non-ergodic measure has a single generic point (see [11, Proposition 9.9]). Moreover, it follows from the ergodic decomposition theorem that even if a non-ergodic measure possesses some generic points, they form a measure zero set. Therefore, an important question is, under what assumptions every invariant measure has a generic point.

E-mail address: martha.ubik@gmail.com.
URL: http://www2.im.uj.edu.pl/MarthaLacka/.

http://dx.doi.org/10.1016/j.jmaa.2017.06.066
0022-247X/© 2017 Elsevier Inc. All rights reserved.

One of conditions which implies such a phenomenon is specification. This property was introduced by Rufus Bowen in [3] to study Axiom A diffeomorphisms. If a dynamical system satisfies specification, then we can find a point that traces an arbitrary collection of orbit segments, if the time between consecutive segments is large enough. Specification implies a very rich dynamics. It is known for instance that it is stronger than chain mixing.

Interestingly, the class of dynamical systems with this property is very wide and contains for example mixing graph maps, mixing sofic shifts (including shifts of finite type) and mixing interval maps. Nevertheless, there are also many interesting systems which shows specification-like behaviour although they do not have the specification property in the sense of Bowen. Therefore many generalizations of this notion have been developed [5,7–9,11,13,12,17,18]. As far as smooth maps are considered, specification is closely related to hyperbolicity.

A dynamical system given by an iteration of a homeomorphism can be regarded as the action of the group of integers on the phase space. The group $\mathbb{Z}$ is an important example of a wide class of amenable residually finite groups (we recall their definitions in the next section).

In the literature one can find some specification-like properties for $\mathbb{Z}^d$-actions or, more generally, amenable groups actions. Among them there are for example topological Markov shifts, strongly irreducible shifts, semi-strongly irreducible shifts [10] or shifts satisfying the uniform filling property [1]. In this paper we use the approach introduced in [4] (see also [14,21]). Recently Ren showed in [20] that if we assume that G is an amenable residually finite group acting on a compact space X and the dynamical systems (X, G) has the specification property in this sense, then the simplex of G-invariant Borel probability measures supported at X is either trivial (that is consists of only one element) or equal to the Poulsen simplex. The latter is a unique (up to an affine homeomorphism) Choquet simplex possessing a dense set of extreme points and has many remarkable properties [16]. Every Choquet simplex can be embedded into the Poulsen simplex as its face. What is more, the set of extreme points of the Poulsen simplex is arcwise connected. The first example of such a simplex was given in [23]. The result of Ren is a generalization of the first part of Sigmund's theorem who showed an analogous claim for actions of the group of integers which satisfy the specification property. The second part of the Sigmund theorem says that in this setting every invariant measure has a generic point.

It is known that even ergodic measures not necessarily have generic points with respect to any Følner sequence. Nevertheless the Lindenstrauss pointwise ergodic theorem guarantees that this is the case if we assume that the Følner sequence is tempered, that is it is growing in a certain way that we describe in more details in the next section. What is important, every amenable group admits a tempered Følner sequence.

We prove that every measure invariant for an amenable residually finite group action satisfying the weak specification property has a generic point. This extends the theorem of Sigmund and completes the result of Ren.

2. General assumptions, definitions and basic facts

2.1. General assumptions and notation

Through (X, ρ) is a compact metric space. To simplify notation we assume that the diameter of X with respect to ρ is equal to 1. An infinite countable group G acts on X via homeomorphisms. We denote by $|A|$ the cardinality of the set A. Given $A, B \subset G$ we define

$$AB = \{ab : a \in A, b \in B\} \text{ and } A^{-1} = \{a^{-1} : a \in A\}.$$

Moreover, we denote by $A \triangle B$ the symmetric difference of A and B. Let $\mathrm{Fin}(G)$ denote the family of finite, non-empty subsets of G. A *fundamental domain* of a finite index subgroup H of a group G is a set F such that for every $g \in G$ we have $|Hg \cap F| = 1$. By $|G : H|$ we denote the index of a subgroup H.

Please cite this article in press as: M. Łącka, Generic points of invariant measures for an amenable residually finite group actions with the weak specification property, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.06.066