J. Math. Anal. Appl. 455 (2017) 1655–1667

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Critical points of orthogonal polynomials

María Pilar Alfaro [a], Manuel Bello-Hernández [b,*,1], Jesús María Montaner [c]

[a] *Departamento de Matemáticas, Universidad de Zaragoza, Calle Pedro Cerbuna s/n, 50009 Zaragoza, Spain*
[b] *Dpto. de Matemáticas y Computación, Universidad de La Rioja, Edif. Científico Tecnológico, Calle Madre de Dios 53, 26006 Logroño, Spain*
[c] *Departamento de Matemática Aplicada, Universidad de Zaragoza, Edif. Torres Quevedo, Calle María de Luna 3, 50018 Zaragoza, Spain*

ARTICLE INFO

Article history:
Received 30 May 2016
Available online 27 June 2017
Submitted by P. Nevai

Keywords:
Orthogonal polynomials
Zeros
Critical points
Asymptotics

ABSTRACT

We study properties of the critical points of orthogonal polynomials with respect to a measure on the unit circle (OPUC). The main result states that, under some conditions, the asymptotic distribution of the critical points of OPUC coincides with the asymptotic distribution of its zeros and each Nevai–Totik point attracts the same number of critical points as zeros of the OPUC. Analogous results are also presented for paraorthogonal polynomials and for orthogonal polynomials with respect to a regular measure supported on a continuum set.

1. Introduction

The critical points of a polynomial are the equilibrium points in a certain force field. The Gauss–Lucas theorem states that the critical points of a polynomial lie in the convex hull of their zeros (see [6, Sect. 2.1]). The Jentzsch–Szegő theorem tells us that for a power series with finite radius of convergence there is an infinite sequence of partial sums, the zeros of which are "equidistributed" with respect to the angular measure on the boundary circle of the disk of convergence. Since a series and its derivative have equal radii of convergence, the result of Jentzsch–Szegő also holds for the corresponding sequence of the derivative. Both theorems have been generalized in different directions (see [11, Sect. 2.1] and [2]). Another interesting open problem in this issue which deserves attention is Sendov's conjecture: "*If p is a polynomial of degree ≥ 2 having all its zeros in the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ and if z_0 is any one of such zero, then at least one critical point of p lies on the disk* $\{z \in \mathbb{C} : |z - z_0| \leq 1\}$" (see [11, Sect. 7.3]). Sendov's conjecture has been proved for polynomials of large degree (see [3]). The asymptotic behavior of the critical

* Corresponding author.
E-mail addresses: palfaro@unizar.es (M.P. Alfaro), mbello@unirioja.es (M. Bello-Hernández), montaner@unizar.es (J.M. Montaner).
1 Supported in part by 'Ministerio de Economía y Competitividad', Project MTM2014-54043-P.

http://dx.doi.org/10.1016/j.jmaa.2017.06.060

points of a sequence of polynomials whose almost all zeros lie in a given convex bounded domain has been proved in [19].

In this paper we study properties of the critical points of orthogonal polynomials with respect to a measure on the unit circle (OPUC). A polynomial whose zeros lie in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is a term of a sequence of polynomials orthogonal with respect to a Bernstein–Szegő measure in the unit circle. Thus, our results want to be in some sense a contribution for a better understanding of the kind of problems above mentioned.

Let μ be a probability measure on $[0, 2\pi)$ and $\{\Phi_n\}_{n\geq 0}$ be the associated sequence of monic OPUC. They satisfy the recurrence relations

$$\begin{cases} \Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \\ \Phi_{n+1}^*(z) = \Phi_n^*(z) + \overline{\Phi_{n+1}(0)}z\Phi_n^*(z), \end{cases} \tag{1}$$

for $n \geq 0$, where $\Phi_0(z) = 1$ and the reverse polynomials are $\Phi_n^*(z) := z^n\overline{\Phi_n(1/\overline{z})}$. As all the zeros of Φ_n are inside $\mathbb{D}$, we have $|\Phi_{n+1}(0)| < 1$ for all $n \geq 0$. According to Verblunsky theorem the map $\mu \mapsto \{\Phi_{n+1}(0)\}_{n\geq 0}$ is a one–one correspondence between non-trivial probability positive Borel measures and sequences in $\mathbb{D}$. Let $\int |\Phi_n(e^{i\theta})|^2\, d\mu(\theta) =: \kappa_n^{-2}$; then, analogous formulae to (1) hold for orthonormal polynomials $\{\varphi_n := \kappa_n\Phi_n\}_{n\geq 0}$. All these results can be found in [13, Chap. 1].

Nevai and Totik [9] proved that if

$$\rho = \limsup_n |\Phi_n(0)|^{1/n} < 1, \tag{2}$$

then $S(z)^{-1}$ can be analytically continued to all the region $\{z \in \mathbb{C} : |z| < 1/\rho\}$, where $S(z)$ represents the Szegő function

$$S(z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log\mu'(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right), \quad z \in \mathbb{D}. \tag{3}$$

This extention $S_{\text{ext}}(z)^{-1}$ has no zeros in $\overline{\mathbb{D}}$. The Nevai–Totik points are the zeros of $S_{\text{ext}}(1/\overline{z})^{-1}$ in $\{z \in \mathbb{C} : \rho < |z| < 1\}$. Since

$$\lim_n \varphi_n^*(z) = S_{\text{ext}}(z)^{-1} \tag{4}$$

holds uniformly on compact subsets of $\{z \in \mathbb{C} : |z| < 1/\rho\}$, the Nevai–Totik points are precisely the limit points of zeros of $\{\varphi_n\}_{n\geq 0}$ in $\{z \in \mathbb{C} : \rho < |z| < 1\}$.

As usual, we say that μ satisfies the Szegő condition or μ belongs to the Szegő class if

$$\int_0^{2\pi} |\log\mu'(\theta)|\,d\theta < \infty.$$

According to Szegő theorem in this case the convergence of $\varphi_n^*(z)$ to $S(z)^{-1}$ is uniform on compact subsets of $\mathbb{D}$.

Let p be a polynomial of degree n and ν_p its normalized counting measure which gives weight k/n to each zero of p with multiplicity k. For $r > 0$, let m_r denote the arc-measure on the circle $C_r := \{z \in \mathbb{C} : |z| = r\}$; i.e. $m_r(\{r\,e^{i\theta} : \theta_1 \leq \theta \leq \theta_2\}) = \theta_2 - \theta_1$ where $0 \leq \theta_1 < \theta_2 < 2\pi$. When $r = 0$, we let m_0 be the delta distribution with mass 1 supported at $z = 0$. If either (2) or