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# A biparametric perturbation method for the Föppl–von Kármán equations of bimodular thin plates

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ABSTRACT

In this study, a biparametric perturbation method is proposed to solve the Föppl–von Kármán equations of bimodular thin plates subjected to a single load. First, by using two small parameters, one describes the bimodular effect and another stands for the central deflection, we expanded the unknown deflection and stress in double power series with respect to the two parameters and obtained the approximate analytical solutions under various edge conditions. Due to the diversity of selection of parameters and its combination, by using the bimodular parameter and the load as two perturbation parameters, we elucidated further the application of this method. The use of two sets of parameter schemes both can obtain satisfactory perturbation solutions; the numerical simulations also verify this idea. The results indicate that in a biparametric perturbation method, the selection and its combination of parameters may reflect the combined effects introduced by nonlinear factors. The method proposed in this study may be used for solving other mathematical equations established in some application problems.



## 1. Introduction

Studies on flexible thin plates may be traced back to the pioneer works performed by Föppl and von Kármán. In 1910, von Kármán established the governing equations describing the large-deflection behavior of a thin flat plate subjected to the external loads [29]

$$D\nabla^4 w = t\left(\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right) + q \tag{1a}$$

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$$\nabla^4\varphi = E\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{1b}$$

where $w = w(x, y)$ is the out-plane deflection, $\varphi = \varphi(x, y)$ is the Airy stress function, $\nabla^4$ denotes the biharmonic operator, $E$ is the Young's modulus of elasticity, $t$ is the thickens of the plate, $q$ is the intensity of the external loads applied, and

$$D = \frac{Et^3}{12(1-\mu^2)} \tag{2}$$

is called the bending stiffness of the plate and $\mu$ is the Poisson's ratio. Eq. (1a) exhibits the relation of equilibrium among the external loads, the tension effect and the bending effect of the plate, while Eq. (1b) indicates the consistency relation between the deflection and the membrane stress. Up to now, the set of Föppl–von Kármán equations has been considered to be an effective mathematical model to analyze the large-deflection deformation of thin flat plates.

During past one centenary, many researchers have applied themselves to the solution of the equations in different ways, including the analytical, numerical and experimental study of these equations (for example, Refs. [7,9,10,26,28], to list but only a few). The problem is often too challenging for analytical methods because the Föppl–von Kármán equations consist of two sets of high-order partial differential equations along with two kinds of deformation. In an axisymmetrical problem, however, the basic variables in the equations, $x$ and $y$, are reduced to only one variable, i.e. the radial coordinates $r$, which permits some researchers to utilize appropriate methods to obtain asymptotic analytical solutions. These methods include the series expansion method in the form of various functions, the variation method based on energy principles and the perturbation method of parameters. Among these methods, the perturbation method seems to be the most attractive one. In the perturbation, the unknown deflection and stress are expanded in the form of ascending powers with respect to a certain small parameter. Substituting the expansions into the governing equations and corresponding boundary conditions will yield a series of equations used for determining the approximate solution of all levels by equating the same order of the perturbation parameter.

Although perturbation solutions may not rely on any small parameter, as suggested by Van Gorder [26], the perturbation parameter plays an important role in the perturbation since the right choice may permit us to obtain asymptotic solutions with better convergence. By using the external load as a perturbation parameter, Vincent [28] firstly obtained the perturbation solution of the Föppl–von Kármán equations. Later Zhou and Zheng [35] analyzed the convergence of Vincent's solution. Given that the perturbation parameter either appears explicitly or is introduced artificially in the problem, Chien [7] obtained another perturbation solution using the central deflection as a perturbation parameter. Comparing with experimental results, Chien's solution is accurate, which has been cited in subsequent studies of the problem for a long period of time. Later, the convergence of Chien's solution is validated by Yeh and Zhou [32]. In addition to the load and the central deflection, there are several other choices for perturbation parameters, for example, a generalized displacement [18], a linear function of Poisson's ratio [24], and an average angular deflection [19]. Chen and Kuang [6] discussed the differences among the possible perturbation parameters and later Chen [5] proposed a free-parameter perturbation method, where the perturbation parameter may have no physical meanings. The proposition of free-parameter perturbation method may be regarded as a supplement to the original perturbation method. More recently, Van Gorder [27] obtained asymptotic solutions for the Föppl–von Kármán equations governing deflections of thin axisymmetric annular plates under different boundary conditions, where the analytical solutions, obtained via a perturbation approach, agreed well with numerical solutions, even for relatively large values of the perturbation parameter.

The above studies were performed only for the Föppl–von Kármán equations established on the isotropic materials model. With the development of the science of materials, the nonlinearity of materials (bimodular materials [20], for example) has been introduced in the study of the classical Föppl–von Kármán equations.