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Sharp approximation formulas and inequalities for the Wallis ratio by continued fraction

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ABSTRACT

In this paper, we present a new continued fraction approximation of the Wallis ratio. This approximation is fast in comparison with the recently discovered asymptotic series. We also establish the inequalities related to this approximation. Finally, some numerical computations are provided for demonstrating the superiority of our approximation.

1. Introduction

The Wallis ratio is defined as

$$W(n) = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)},$$

where Γ is the classical Euler gamma function which may be defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \mathrm{Re}(x) > 0.$$

The study and applications of $W(n)$ have a long history, a large amount of literature, and a lot of new results. For detailed information, please refer these papers [4,5,10,12] and references cited therein.

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Chen and Qi [4] presented the following inequalities for the Wallis ratio for every natural number n:

$$\frac{1}{\sqrt{\pi(n+\frac{4}{\pi}-1)}} \le \frac{(2n-1)!!}{(2n)!!} \le \frac{1}{\sqrt{\pi(n+\frac{1}{4})}}, \tag{1.1}$$

where the constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible.

Guo, Xu and Qi proved in [6] that the double inequality

$$\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n} < W(n) \le \frac{4}{3}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}, \tag{1.2}$$

for $n \ge 2$ is valid and sharp in the sense that the constants $\sqrt{\frac{e}{\pi}}$ and $\frac{4}{3}$ are best possible. They also proposed the approximation formula

$$W(n) \sim \chi(n) := \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n}, n \to \infty. \tag{1.3}$$

Recently, Qi and Mortici [11] improved the approximation formula (1.3) as following,

$$W(n) \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{1}{\sqrt{n}} \exp\left(\frac{1}{24n^2}+\frac{1}{48n^3}+\frac{1}{160n^4}+\frac{1}{960n^5}+\cdots\right), n \to \infty. \tag{1.4}$$

Motivated by these works, in this paper we will apply the *multiple-correction method* [1–3] to construct a new asymptotic expansion for the Wallis ratio by continued fraction as follows:

Theorem 1. *For the Wallis ratio $W(n) = \frac{(2n-1)!!}{(2n)!!}$, we have*

$$W(n) \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{1}{\sqrt{n}} \exp\left(\frac{\frac{1}{24}}{n^2-\frac{1}{2}n+\frac{1}{10}+\frac{-\frac{41}{1400}}{n^2-\frac{1}{2}n+\frac{16}{15}+\frac{-\frac{194815}{227304}}{n^2-\frac{1}{2}n+\frac{9269629}{3039114}}}}\right). \tag{1.5}$$

Using Theorem 1, we provide some inequalities for the Wallis ratio.

Theorem 2. *For every integer $n > 1$, it holds:*

$$\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{1}{\sqrt{n}} \exp\left(\frac{\frac{1}{24}}{n^2-\frac{1}{2}n+\frac{1}{10}+\frac{-\frac{41}{1400}}{n^2-\frac{1}{2}n+\frac{16}{15}+\frac{-\frac{194815}{227304}}{n^2-\frac{1}{2}n+\frac{9269629}{3039114}}}}\right) \tag{1.6}$$

$$< W(n) < \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{1}{\sqrt{n}} \exp\left(\frac{\frac{1}{24}}{n^2-\frac{1}{2}n+\frac{1}{10}+\frac{-\frac{41}{1400}}{n^2-\frac{1}{2}n}}\right).$$

To obtain Theorem 1, we need the following lemma which was used in [7–9] and is very useful for constructing asymptotic expansions.

Lemma 1. *If the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to zero and there exists the limit*

$$\lim_{n\to+\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty] \tag{1.7}$$