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Uniqueness of positive solutions to some nonlinear Neumann problems ☆

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A B S T R A C T

Using the moving plane method, we obtain a Liouville type theorem for nonnegative solutions of the Neumann problem

$$\begin{cases} \operatorname{div}\left(y^a \nabla u(x,y)\right) = 0, & x \in \mathbb{R}^n, y > 0, \\ \lim\limits_{y\to 0+} y^a u_y(x,y) = -f(u(x,0)), & x \in \mathbb{R}^n, \end{cases}$$

under general nonlinearity assumptions on the function $f : \mathbb{R} \to \mathbb{R}$ for any constant $a \in (-1, 1)$.

Contents

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1. Introduction and main results

1.1. Introduction

Let $a \in (-1,1)$, $n \geq 1$ and $\mathbb{H}$ denote the upper half space

$$\mathbb{H} = \{(x,y) : x \in \mathbb{R}^n, y > 0\}.$$

In this paper, we consider the Neumann problem

$$\begin{cases} \operatorname{div}(y^a \nabla u(x,y)) = 0 & \text{in } \mathbb{H}, \\ \dfrac{\partial u}{\partial \nu^a} = f(u) & \text{on } \partial\mathbb{H}, \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \to [0,\infty)$ is a nonnegative function, $\nabla = (\partial_{x_1}, \cdots, \partial_{x_n}, \partial_y)$ is the full gradient operator in $\mathbb{H}$, and

$$\frac{\partial u}{\partial \nu^a} = -\lim_{y \to 0+} y^a \partial_y u(x,y).$$

Equation (1.1) has been studied extensively in the literature. Indeed, equation (1.1) is closely related to the fractional Laplacian equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

where $(-\Delta)^s$ is the usual fractional Laplacian operator defined via its multiplier $|\xi|^{2s}$ in Fourier space, and we denote

$$s = (1-a)/2$$

throughout the paper. This connection has been highlighted by Caffarelli and Silvestre [4] and by related applications such as Cabré and Sire [2,3], Frank and Lenzmann [15] and Frank et al. [16]. More precisely, let $y > 0$ and let $P_y^a : \mathbb{R}^n \to \mathbb{R}$ be the kernel given by

$$P_y^a(x) = k_a y^{-n} \left(1 + (|x|/y)^2\right)^{-(n+1-a)/2}, \quad x \in \mathbb{R}^n,$$

where the positive constant k_a is chosen such that $\int_{\mathbb{R}^n} P_y^a(x)\mathrm{d}x = 1$. It was proven in Caffarelli and Silvestre [4] that for sufficiently regular function ϕ in $\mathbb{R}^n$ (e.g., ϕ belongs to the fractional Sobolev space $\dot{H}^s(\mathbb{R}^n)$), the function $\Phi : \mathbb{H} \to \mathbb{R}$ defined as

$$\Phi(x,y) = P_y^a * \phi(x) = \int_{\mathbb{R}^n} P_y^a(x-z)\phi(z)\mathrm{d}z, \quad (x,y) \in \mathbb{H},$$

is an extension of ϕ to the upper half plane, such that $\lim_{y \to 0+} \Phi(x,y) = \phi(x)$ holds on $\partial\mathbb{H}$ in some sense. Moreover, Φ solves the boundary problem

$$\begin{cases} \operatorname{div}(y^a \nabla \Phi(x,y)) = 0, & (x,y) \in \mathbb{H}, \\ \partial\Phi/\partial\nu^a = d_s(-\Delta)^s \phi, & \text{on } \partial\mathbb{H}, \end{cases}$$