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Topological properties of self-similar fractals with one parameter ☆

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ABSTRACT

In this paper, we study two classes of planar self-similar fractals T_ε with a shifting parameter ε. The first one is a class of self-similar tiles by shifting x-coordinates of some digits. We give a detailed discussion on the disk-likeness (*i.e., the property of being a topological disk*) in terms of ε. We also prove that T_ε determines a quasi-periodic tiling if and only if ε is rational. The second one is a class of self-similar sets by shifting diagonal digits. We give a necessary and sufficient condition for T_ε to be connected.

1. Introduction

Let A be a $d \times d$ integer expanding matrix (i.e., all of its eigenvalues are strictly larger than one in modulus), let $\mathcal{D} = \{d_1, \dots, d_N\} \subset \mathbb{R}^d$ be a finite set of vectors for some integer $N > 1$, we term it a *digit set*. Let $\{S_j\}_{j=1}^N$ be an *iterated function system (IFS)* where $S_j(x) = A^{-1}(x + d_j),\ j = 1, \dots, N$ are affine maps. Since A is expanding, each S_j is a contractive map under a suitable norm [12] on $\mathbb{R}^d$, there is a unique nonempty compact subset $T := T(A, \mathcal{D}) \subset \mathbb{R}^d$ [9] such that

$$T = \bigcup_{j=1}^{N} S_j(T) = A^{-1}(T + \mathcal{D}).$$

Usually T is given explicitly by

$$T = \left\{ \sum_{k=1}^{\infty} A^{-k} d_{j_k} : d_{j_k} \in \mathcal{D} \right\}. \tag{1.1}$$

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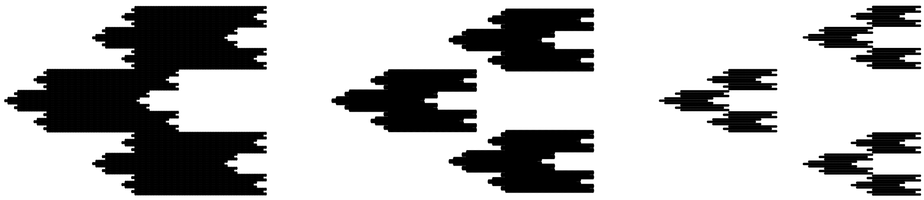

(a) $\varepsilon = 2$ (b) $\varepsilon = 4$ (c) $\varepsilon = 9$

Fig. 1. An illustration of Theorem 1.1 by taking $p = 3$.

We call T a *self-affine set* generated by the pair $(A, \mathcal{D})$ (or the IFS $\{S_j\}_{j=1}^N$). If, in addition, $N = |\det(A)|$ and T has non-void interior (i.e., $T^\circ \neq \emptyset$), then we call T a *self-affine tile*. That is, there exists a set $\mathcal{J} \subset \mathbb{R}^d$ such that

$$T + \mathcal{J} = \mathbb{R}^d \quad \text{and} \quad (T^\circ + t) \cap (T^\circ + t') = \emptyset \quad \text{with} \quad t \neq t' \in \mathcal{J}.$$

In this case, $T + \mathcal{J}$ is named a *tiling* of $\mathbb{R}^d$. Particularly, if A is a constant multiple of an orthonormal matrix (i.e., A is a similarity and the $\{S_j\}_{j=1}^N$ are similitudes), then T is called a *self-similar set/tile* accordingly.

Since the fundamental theory of self-affine tiles was established by Lagarias and Wang ([12], [13], [14]), there have been considerable interests in the topological structure of self-affine tiles T, including but not limited to the connectedness of T ([6], [7], [11], [1], [5], [20]), the boundary ∂T ([2], [17], [22]), or the interior T° of a connected tile T ([24], [25]). Especially in $\mathbb{R}^2$, the study on the disk-likeness of T (*i.e., the property of being a topological disk*) has attracted a lot of attentions ([4], [15], [23], [10], [5]). For other works on self-affine sets, we refer to [16], [18], [19], [21].

Any change on the matrix A and the digit set $\mathcal{D}$ may lead to some change on the topology of $T(A, \mathcal{D})$. To simplify the analysis on the relations between those two types of "changes", one may fix an expanding matrix A and focus on particular choices of the digit set $\mathcal{D}$. Recently Deng and Lau [5] considered a class of planar self-affine tiles T that are generated by a lower triangular expanding matrix and product-form digit sets. They gave a complete characterization on both connectedness and disk-likeness of T.

Motivated by the above results, in this paper, we investigate the topological properties of the following two classes of self-similar fractals in $\mathbb{R}^2$. Assume that A is a diagonal matrix with equal nonzero entries, hence A is a similarity. In the first class, we consider a kind of digit sets $\mathcal{D}_\varepsilon$ with a shift ε on the x-coordinates of some digits. We obtain an analogous result to [5].

Theorem 1.1. *Let p be an integer with $|p| = 2m + 1$ where $m \in \mathbb{N}$, let $\varepsilon \in \mathbb{R}$. Suppose T_ε is the self-similar set generated by* $A = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$ *and*

$$\mathcal{D}_\varepsilon = \left\{ \begin{bmatrix} i + b_j \\ j \end{bmatrix} : b_j = \frac{1 - (-1)^j}{2}\varepsilon, \quad i, j \in \{0, \pm 1, \dots, \pm m\} \right\}.$$

Then T_ε is a self-similar tile. Moreover,

(i) if $|\varepsilon| < |p|$, then T_ε is disk-like;

(ii) if $|p|^n \leq |\varepsilon| < |p|^{n+1}$ for $n \geq 1$, then T_ε° has $|p|^n$ components and the closure of every component is disk-like. (See Fig. 1.)

Furthermore, we consider the quasi-periodic tiling property of T_ε (the definition will be recalled in Section 3). Let $\mathcal{D}_{\varepsilon,k} = \mathcal{D}_\varepsilon + A\mathcal{D}_\varepsilon + \cdots + A^{k-1}\mathcal{D}_\varepsilon$ and $\mathcal{D}_{\varepsilon,\infty} = \bigcup_{k=1}^\infty \mathcal{D}_{\varepsilon,k}$. We prove that