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A characterization of well-posedness for abstract Cauchy problems with finite delay

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ABSTRACT

Let A be a closed operator defined on a Banach space X and F be a bounded operator defined on a appropriate phase space. In this paper, we characterize the well-posedness of the first order abstract Cauchy problem with finite delay,

$$\begin{cases} u'(t) = Au(t) + Fu_t, & t > 0; \\ u(0) = x; & \\ u(t) = \phi(t), & -r \leq t < 0 \end{cases},$$

solely in terms of a strongly continuous one-parameter family $\{G(t)\}_{t\geq 0}$ of bounded linear operators that satisfy the functional equation

$$G(t+s)x = G(t)G(s)x + \int_{-r}^{0} G(t+m)[SG(s+\cdot)x](m)dm$$

for all $t, s \geq 0,\ x \in X$. In case $F \equiv 0$ this property reduces to the characterization of well-posedness for the first order abstract Cauchy problem in terms of the functional equation that satisfy the C_0-semigroup generated by A.

1. Introduction

Let X be a complex Banach space. In this paper, we study the first order abstract Cauchy problem with finite delay

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$$\begin{cases} u'(t) = Au(t) + Fu_t, & t > 0; \\ u(0) = x; & \\ u(t) = \phi(t), & -r \le t < 0 \end{cases}, \tag{1.1}$$

where A is a closed linear operator with domain $D(A)$, $F : L^p([-r, 0], X) \to X$ is a bounded linear map, r is a positive number and ϕ is a given initial function.

The field of linear (and nonlinear) delay differential equations has undergone a significant development for several decades. In the last years, its interaction with other scientific fields has also increasing interest, in particular, in the study of biological models.

In case $F \equiv 0$ it is well known that (1.1) is well-posed if and only if A is the generator of a C_0-semigroup, that is, there exists a strongly continuous family of bounded and linear operators $\{T(t)\}_{t\ge 0}$ satisfying $T(0) = I$ and the Cauchy's functional equation

$$T(t + s)x = T(t)T(s)x, \quad t, s \ge 0, \quad x \in X. \tag{1.2}$$

The theory of C_0-semigroups is a well-established and developed theory, that in some extent began with the original monograph of Hille and Phillips [9]. For an up to date reference and historical remarks, see e.g. Engel and Nagel [6].

In case $F \neq 0$ there is an important amount of literature on the subject. For instance, Hale [8] and Webb [20] began an abstract analysis, i.e. in the setting of Banach spaces, applying methods coming from semigroup theory. After that, Travis and Webb [18, Section 4] studied existence and stability of solutions when A is the generator of a compact semigroup, or an analytic semigroup [19]. Fitzgibbon [7] was among the first to treat the nonautonomous case i.e. $A = A(t)$. Jiang, Guo and Huang [10] studied the well-posedness of the linear abstract problem with unbounded delay operators. More recently, Ashyralyev and Agirseven [3] analyzed the well-posedness of (1.1) when the delay admits the form of a nonautonomous and unbounded operator.

After the method of semigroups, most of the approaches consists into associate to a given delay equation an expanded space E (phase space) and a lifted unbounded operator $(B, D(B))$ and to demonstrate that the solutions of the abstract Cauchy problem associated to $(B, D(B))$ in E naturally correspond to those of the delay equation. Then, the task is to show that the lifted operator $(B, D(B))$ generates a strongly continuous semigroup $\{T(t)\}_{t\ge 0}$ on E, thus implying the Cauchy problem is well-posed. See e.g. [5] and the monograph of Bátkai and Piazzera [4].

However, this last method produces significant mathematical difficulties when we deal with e.g. the regularity problem. For instance, suppose that the operator A in (1.1) generates an analytic semigroup, a condition which is frequently assumed in the investigation of the regularity problem. Then the lifted generator $(B, D(B))$ of the system does not generate an analytic semigroup any more on the expanded spaces (cf. [11]).

First attempts to treat directly (1.1), that is without any assumption on the operator A and neither appealing to some phase space, were made by Petzeltová [16], [17]. By replacing X with a suitable interpolation space, she proves the existence of a family of bounded and linear operators $R(t)$ satisfying $R'(t) = AR(t) + FR_t, R(0) = I, R_0 = 0$.

In a recent paper, Liu [12] employed a direct method to deal with the regularity problem. Liu developed a theory of retarded type operators $\{G(t)\}$, or fundamental solutions, for (1.1) defined in a Hilbert space H. Among other interesting things, Liu proves that the following functional equation is satisfied:

$$G(t + s)x = G(t)G(s)x + \int_{-r}^{0} G(t + m)[SG(s + \cdot)x](m)dm, \ \ t, s \ge 0, \ x \in H, \tag{1.3}$$