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An extension of Borel–Laplace methods and monomial summability [☆]

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In this paper we will show that monomial summability can be characterized using Borel–Laplace like integral transformations depending of a parameter $0 < s < 1$. We will apply this result to prove 1-summability in a monomial of formal solutions of a family of partial differential equations.

1. Introduction

Monomial summability was developed [4] in order to understand summability properties of formal solutions of a class of singularly perturbed differential systems (doubly singular differential systems). In contrast with the classical theory of summability, there was not available an approach to the concept using integral transformation, i.e. a Borel–Laplace method of summation. On the other hand, in [3] these formal solutions were studied in the linear case using certain integral transformations, also introduced later in [2] for any number of variables, but the relation between the methods of summation was not clear. It turns out that using an adaptation of those operators the problem of understanding that relation can be solved. In fact, one of the main results in this work is the following one (appropriate definitions will be introduced later):

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Theorem 4.1. *Let $\hat{f}$ be a $1/k$-Gevrey series in the monomial $x_1^p x_2^q$. Then it is equivalent:*

(1) $\hat{f} \in E\{x_1, x_2\}_{1/k,d}^{(p,q)}$, i.e. $\hat{f}$ is k-summable in $x_1^p x_2^q$ in direction d.
(2) There is $0 < s < 1$ such that $\hat{f}$ is $k - (s, 1-s)$-Borel summable in the monomial $x_1^p x_2^q$ in direction d.
(3) For all $0 < s < 1$, $\hat{f}$ is $k - (s, 1-s)$-Borel summable in the monomial $x_1^p x_2^q$ in direction d.

In all cases the corresponding sums coincide.

This result shows in particular that the integral methods introduced in [3] agree with the notion of monomial summability. And, moreover, they can be used in order to treat more complicated systems, for instance, of partial differential equations, namely, families of PDEs as follows:

$$x_1^p x_2^q \left(\frac{s}{p} x_1 \frac{\partial \mathbf{y}}{\partial x_1} + \frac{1-s}{q} x_2 \frac{\partial \mathbf{y}}{\partial x_2} \right) = C(x_1, x_2)\mathbf{y} + \gamma(x_1, x_2).$$

These systems generalize the ones in [4] for $s = 0$ and $s = 1$, under the assumption that $C(0,0)$ is invertible.

The plan of the paper is as follows: in Section 2, the notions of asymptotic expansions and summability, both in the one variable, and in the monomial case, are reviewed. Monomial Borel and Laplace transforms are defined and investigated in Section 3, in order to introduce the summation methods using these operator in Section 4. The main result of this work, Theorem 4.1, is stated and proved here. Finally, Section 5 provides examples of application to a family of partial differential equations. A possible further development, in order to define multisummability in the monomial case, is sketched in Section 6.

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2. Summability

We recall briefly the basic notions of asymptotic expansions, Gevrey asymptotic expansions and summability in the case of one variable and their extensions to the monomial case in two variables, as introduced in the paper [4].

Let $(E, \|\cdot\|)$ be a complex Banach space and $\hat{f} = \sum a_n x^n \in E[[x]]$. We will denote by $\mathcal{C}(U, E)$ (resp. $\mathcal{O}(U, E)$, $\mathcal{O}_b(U, E)$) the space of continuous E-valued maps (resp. holomorphic, holomorphic and bounded E-valued maps) defined on an open set $U \subset \mathbb{C}^l$. If $E = \mathbb{C}$ we will simply write $\mathcal{O}(U)$. We also denote by $\mathbb{N}$ the set of natural numbers including 0, by $\mathbb{N}_{>0} = \mathbb{N} \setminus \{0\}$, by $D_r \subset \mathbb{C}$ the disk centered at the origin with radius r and by $V = V(a, b, r) = \{x \in \mathbb{C} | 0 < |x| < r, a < \arg(x) < b\}$ an open sector in $\mathbb{C}$. When we want to emphasize the bisecting direction $d = (b + a)/2$ and the opening of the sector we will write $V = S(d, b-a, r) = S$. In the case $r = +\infty$ we will simply write $V(a, b) = S(d, b-a)$. Consider $f \in \mathcal{O}(V, E)$. The map f is said to have $\hat{f}$ as *asymptotic expansion* at the origin on V (denoted by $f \sim \hat{f}$ on V) if for each of its proper subsectors $V' = V(a', b', r')$ ($a < a' < b' < b$, $0 < r' < r$) and each $N \in \mathbb{N}$, there exists $C_N(V') > 0$ such that

$$\left\| f(x) - \sum_{n=0}^{N-1} a_n x^n \right\| \leq C_N(V') |x|^N, \quad \text{on } V'. \tag{1}$$