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On the longest block in Lüroth expansion

Jinjun Li [a,*], Min Wu [b], Xiangfeng Yang [c]

[a] School of Mathematics and Information Science, Guangzhou University, Guangzhou, 510006, PR China
[b] Department of Mathematics, South China University of Technology, Guangzhou, 510641, PR China
[c] Department of Mathematics, Linköping University, Linköping, SE-581 83, Sweden

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ABSTRACT

In this paper, for a finite subset $A \subset \{2, 3, \cdots\}$, we introduce the notion of longest block function $L_n(x, A)$ for the Lüroth expansion of $x \in [0, 1)$ with respect to A and consider the asymptotic behavior of $L_n(x, A)$ as n tends to ∞. We also obtain the Hausdorff dimensions of the level sets and exceptional set arising from the longest block function.

1. Introduction

It is well known that each $x \in [0, 1)$ can be written as a finite or infinite series in the following form:

$$x = \frac{1}{a_1(x)} + \sum_{n\geq 2} \frac{1}{a_1(x)(a_1(x)-1)\cdots a_{n-1}(x)(a_{n-1}(x)-1)a_n(x)}, \tag{1}$$

where $a_n(x)(n \geq 1)$ are appropriately defined integers not less than 2. Such a series is called the Lüroth expansion of x, which was introduced in 1883 by Lüroth [15], and denoted as

$$x = [a_1(x), a_2(x), \cdots, a_n(x), \cdots].$$

In the view of dynamical systems, the series in (1) is closely related to the dynamics of the Lüroth map $T : [0, 1) \to [0, 1)$ defined by

* Corresponding author.
E-mail addresses: li-jinjun@163.com (J. Li), wumin@scut.edu.cn (M. Wu), xiangfeng.yang@liu.se (X. Yang).

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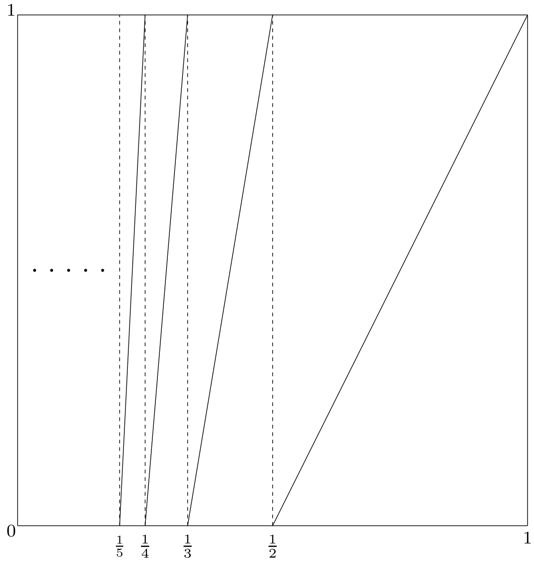

Fig. 1. The Lüroth map T.

$$T(x) = \begin{cases} n(n+1)x - n, & x \in [\frac{1}{n+1}, \frac{1}{n}), \\ 0, & x = 0. \end{cases}$$

More precisely, the digits $a_n = a_n(x)$ in (1) are defined by

$$a_n(x) = a_1(T^{n-1}x),$$

where $a_1(x) = n$ if $x \in [\frac{1}{n}, \frac{1}{n-1})$, $n \geq 2$. Fig. 1 illustrates the Lüroth map T.

The digits $\{a_n(x), n \geq 1\}$ can be regarded as random variables on the measure space $([0,1), \mathcal{B}([0,1)), \mathcal{L})$, where $\mathcal{B}([0,1))$ is the σ-algebra of Borel subsets of $[0,1)$ and $\mathcal{L}$ is the Lebesgue measure on $[0,1)$. Moreover, they are independent with the same distribution as follows

$$l_k := \mathcal{L}(\{a_n(x) = k\}) = \frac{1}{k(k-1)}, \quad k \geq 2,$$

see Theorem 4.14 in [9]. It is interesting to study the statistical properties of the digits $\{a_n, n \geq 1\}$ because many important sets in $[0,1)$ can be characterized by or are closely related to some special properties of the digits. For example, each irrational number has a unique infinite Lüroth expansion and each rational number has either a finite expansion or a periodic one; see [15] or [5]. Some kinds of frequencies of the digits $\{a_n, n \geq 1\}$ were studied in [2,8,10]. The asymptotic behavior of the largest digit among the first n terms in the Lüroth expansion was studied in [15] and [19]. Moreover, many fractal sets related to the digits of Lüroth expansion were studied as well, see [18,23] and the references therein.

In this paper, we shall study the asymptotic behavior of another quantity related to the digits $\{a_n, n \geq 1\}$. Let $A \subset \{2, 3, \cdots\}$ be a finite subset. For $x \in [0,1)$ and $n \in \mathbb{N}$, we define the longest block function $L_n(x) = L_n(x, A)$ with respect to A as the length of the longest consecutive sequence whose elements are all in A during the first n digits of x. Namely,

$$L_n(x) = \max\{\ell : a_{i+1} \in A, a_{i+2} \in A, \cdots, a_{i+\ell} \in A \text{ for some } 0 \leq i \leq n - \ell\}, \tag{2}$$

where $x = [a_1(x), a_2(x), \cdots, a_n(x), \cdots]$.

We have the following asymptotic behavior of $L_n(x)$.