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Dynamics of pulse solutions in Gierer–Meinhardt model with time dependent diffusivity

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ABSTRACT

Dispersive processes with a time dependent diffusivity appear in a plethora of physical systems. Most often a solution is attained for a predefined form of diffusion coefficient $D(t)$. Here existence of pulse solutions with an arbitrary time dependence thereof is proved for the Gierer–Meinhardt model with three types of transport: regular diffusion, sub-diffusion and Lévy flights. Admission of a solution of the classical pulse shape, but for an unencumbered form of $D(t)$ is a valuable property that allows to study phenomena of the ilk observed in various ostensibly unrelated applications. Closed form solutions are obtained for some pulse constellations. Transitions between periods of nearly constant diffusivities trigger respective cross-over between counterpart solutions known for a constant diffusivity, thereupon exhibiting otherwise unattainable behaviour, qualitatively reconstructing observable evolution peculiarities of tagged molecular structures, such as essential slowing down or speeding up during various stages of motion, inexplicable with a single constant diffusion coefficient.

1. Background

Processes with a time dependent diffusivity are used to model multifarious physical systems. Biological media often evince a non-linear dependence of the mean square displacement in time, especially in measurements of short duration [23]. Although conventionally the diffusivity is deemed an essential quality of the diffusing substance, in complex media the influence of the environment is such that dispersive properties at a given moment in time depend on past evolution, endowing the diffusion coefficient with temporal dependence, often an algebraically decaying function $D(t) \sim t^{-\alpha}$, $0 < \alpha < 1$. A process, whose purport is to deliver ligands, might naturally be punctuated by ingress into organelles, egress therefrom, passage through a highly crowded environment or one of a different effective dimension. An instance thereof is when molecules cross from the cytoplasm into a membrane, instantly changing a bulk diffusion, possibly in a domain of a fractional dimension, into lateral diffusion. Such transitions result in further alteration of the diffusivity

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function and have been accorded some attention, cf. [1], [2], [17], [18] and [24]. An entirely distinct type of process is observed in flows through porous media. There the environment's intricate geometry renders the diffusion coefficient time dependent, whilst there exists no particular moment in time that can be designated as a cross-over and attributed to a physical change. The diffusivity decays and saturates smoothly, and might be approximated by combinations of powers, the precise form depending on the choice of packing and other geometric parameters [10]. Similar observations are reported in [7] and [8]. Between these two extremities lie systems with blended characteristics, cf. [3,9] and [25]. What unifies all these processes is the memory, i.e. dependence of dispersive properties on past evolution.

Following the cue of foregoing theoretical and experimental studies, the current contribution seeks to study transitions in a generic pattern forming system. Analysis of this ilk is expected to put forth qualitative insight into the transition mechanism and in particular the fundamental question whether based on the macroscopic measure of the mean square displacement it is possible to distinguish between transitions in the diffusant's molecular properties and those of the medium. The system chosen must be sufficiently explored with constant D, exhibit flexibility of generalisation to diffusive regimes beyond regular Fickian diffusion and be amenable to solution with a wide class of functions $D(t)$.

In this regard the Gierer–Meinhardt model is an apposite choice. It is a chemistry based reaction–diffusion system, forming patterns as a result of the interplay between two components, activator and inhibitor [4]. At the limit of an asymptotically small activator to inhibitor diffusivity ratio the system possesses fully non-linear solutions in the shape of localised structures known as spikes or pulses, and has been extensively studied, cf. [15], [22] and references therein. Notably this system evinces a unique diversity of possible patterns, with multiple spikes of virtually arbitrary loci [5]. Of particular interest proved to be patterns entailing symmetry [6] and certain classes of asymmetry [20]. Furthermore, subsequent studies analysed the system with fractional operators, corresponding to sub-diffusion [13] and Lévy flights [11]. The existence of spike patterns with anomalous diffusion thus established the Gierer–Meinhardt system as a powerful paradigmatic pattern formation model: the mechanism responsible for the emergence of fully non-linear structures is so fundamental that the pattern persists even under significant modifications of the transport properties.

The current contribution proves the existence of a spike pattern for the Gierer–Meinhardt system with a generic time dependent diffusion coefficient. The theory is used to exemplify the pattern evolution subject to a transition in the diffusivity and impact thereof when superimposed on an inherently varying diffusivity, as in sub-diffusion and Lévy flights types of transport. Simulation of the spike drift within the domain qualitatively reconstructs the observations referred to above of slowed down or sped up progress of tagged molecular clusters, once more attesting to the dramatic structural strength of the Gierer–Meinhardt system as a phenomenological model.

The mainstay of the analysis regards the construction of a spike solution in the presence of an arbitrary diffusion coefficient function that retains the unity order of magnitude throughout the evolution process. This feature is very powerful since it gives absolute freedom to explore new types of diffusivity functions of interest if such be identified in the future. The pattern of spikes exists as a quasi-equilibrium solution, whose slow drift is governed by a differential–algebraic system of non-linear equations. §2 derives this system with regular diffusion, generalising a former result with constant diffusivity [5]. Thereafter §3 and §4 further extend the analysis to sub-diffusive transport and Lévy flights respectively.

2. Spike pattern: regular diffusion

Diffusion and reaction of two species, activator $a(x,t)$ and inhibitor $h(x,t)$ within a one dimensional finite domain is governed by

$$\partial_t a = \epsilon^2 \partial_x^2 a - a + \frac{a^p}{h^q}, \quad -1 < x < 1,\ t > 0, \tag{1a}$$