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On weighted conditions for the absolute convergence of Fourier integrals

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ABSTRACT

In this paper we obtain new sufficient conditions for representation of a function as an absolutely convergent Fourier integral. Unlike those known earlier, these conditions are given in terms of belonging to weighted spaces. Adding weights allows one to extend the range of application of such results to Fourier multipliers with unbounded derivatives.

1. Introduction

If, for $n = 1, 2, \dots$

$$f(x) = \int_{\mathbb{R}^n} g(t) e^{i(x,t)} dt, \qquad g \in L_1(\mathbb{R}^n), \tag{1.1}$$

where $(x,t) = x_1t_1 + \dots + x_nt_n$, we say that f belongs to Wiener's algebra $W_0(\mathbb{R}^n)$, written $f \in W_0(\mathbb{R}^n)$, with $\|f\|_{W_0} = \|g\|_{L_1(\mathbb{R}^n)}$. Wiener's algebra is an important class of functions and its in-depth study is motivated both by many points of interest in the topic itself and by its relations to other areas of analysis, such as Fourier multipliers or comparison of differential operators. The history, motivations and various conditions of belonging to Wiener's algebra are overviewed in detail in a recent survey paper [13] (see also [30, Ch.6]).

Of course, [13] summarized the long term studies of many mathematicians and gave a comprehensive picture of the subject. However, these studies are continuing, see, e.g., [9–15,29]. In these works the under-

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taken efforts have mainly been aimed at obtaining conditions of mixed type, in the sense that conditions are posed simultaneously on the function and its derivatives.

Naturally, certain such conditions were known earlier, see, e.g., [3,16]. In the latter, the (multidimensional) Riesz fractional differentiation is defined by

$$(-\Delta)^{\frac{\alpha}{2}} f = \mathscr{F}^{-1}|x|^{\alpha}\mathscr{F} f, \qquad \alpha > 0,$$

where $\mathscr{F}$ means the Fourier operator, while $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ denotes the Laplace operator, and the result reads as follows (see [16]).

Theorem A. *Let $f \in L_2(\mathbb{R}^n)$, and $(-\Delta)^{\frac{\alpha}{2}} f \in L_2(\mathbb{R}^n)$, $\alpha > \frac{n}{2}$, then $\mathscr{F} f \in L_1(\mathbb{R}^n)$.*

To give further convenient formulations in the multivariate case, we need additional notations. Let η be an n-dimensional vector with the entries either 0 or 1 only. Here and in what follows, $D^{\eta} f$ for $\eta = \mathbf{0} = (0, 0, ..., 0)$ or $\eta = \mathbf{1} = (1, 1, ..., 1)$ mean the function itself and the mixed derivative in each variable, respectively, where

$$D^{\eta} f(x) = \left(\prod_{j:\, \eta_j = 1} \frac{\partial}{\partial x_j} \right) f(x).$$

One of the multidimensional results we are going to generalize reads as follows (see [24]).

Theorem B. *Let $f \in L_1(\mathbb{R}^n)$. If all the mixed derivatives (in the distributional sense) $D^{\eta} f(x) \in L_p(\mathbb{R}^n)$, $\eta \neq \mathbf{0}$, where $1 < p \le 2$, then $f \in W_0(\mathbb{R}^n)$.*

However, the main motivation to our work was given by the following recent result [11]: if $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, and $f' \in L^q(\mathbb{R})$, $1 < q < \infty$, for p and q such that

$$\frac{1}{p} + \frac{1}{q} > 1, \tag{1.2}$$

then $f \in W_0(\mathbb{R})$; and if $\frac{1}{p} + \frac{1}{q} < 1$, then there is a function f such that $f \in L^p(\mathbb{R})$, $f' \in L^q(\mathbb{R})$ and $f \notin W_0(\mathbb{R})$. Certainly, this result essentially generalizes many previous results, e.g., the one in [3], but we highlight it not only as a "landmark" but also since we shall pay much attention to the case where $\frac{1}{p} + \frac{1}{q} = 1$ and discuss why it does not ensure the belonging to W_0, except a unique special case $p = q = 2$ (see the proof of Proposition 6.3). Despite the attraction of (1.2), one could see already in [11] the incompleteness of this condition. Indeed, it assumes not only the function to be "good" but the derivative to be "good" as well. However, related results on Fourier multipliers show that it is by no means necessary. The model case is delivered by the well-known multiplier (see [8], [26, Ch.4, 7.4], [5])

$$m(x) = m_{\alpha,\,\beta}(x) = \rho(x) \frac{e^{i|x|^{\alpha}}}{|x|^{\beta}}, \tag{1.3}$$

where ρ is a C^{∞} function on $\mathbb{R}^n$, vanishing for $|x| \le 1$ and equal to 1 if $|x| \ge 2$, with $\alpha, \beta > 0$. Recall that the Fourier multipliers are defined as follows. Let $m : \mathbb{R}^n \to \mathbb{C}$ be an almost everywhere bounded measurable function ($m \in L_{\infty}(\mathbb{R}^n)$). Define on $L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ a linear operator Λ by means of the following identity for the Fourier transforms of functions $f \in L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$:

$$\mathscr{F}(\Lambda f)(x) = m(x) \mathscr{F} f(x).$$

If a constant $D > 0$ exists such that for each $f \in L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ there holds